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A General Framework of Compactly Supported Splines and Wavelets*

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Let $\{V_k\}$ be a nested sequence of closed subspaces that constitute a multiresolution analysis of $L^2(\mathbb{R})$. We characterize the family $\Phi = \{\phi\}$ where each ϕ generates this multiresolution analysis such that the two-scale relation of ϕ is governed by a finite sequence. In particular, we identify the $\varphi \in \Phi$ that has minimum support. We also characterize the collection Ψ of functions η such that each η generates the orthogonal complementary subspaces W_k of V_k, $k \in \mathbb{Z}$. In particular, the minimally supported $\psi \in \Psi$ is determined. Hence, the "B-spline" and "B-wavelet" pair (φ, ψ) provides the most economical and computational efficient "spline" representations and "wavelet" decompositions of L^2 functions from the "spline" spaces V_k and "wavelet" spaces W_k, $k \in \mathbb{Z}$. A very general duality principle, which yields the dual bases of both $\{\varphi(\cdot - j): j \in \mathbb{Z}\}$ and $\{\eta(\cdot - j): j \in \mathbb{Z}\}$ for any $\eta \in \Psi$ by essentially interchanging the pair of two-scale sequences with the pair of decomposition sequences, is also established. For many filtering applications, it is very important to select a multiresolution for which both φ and ψ have linear phases. Hence, "non-symmetric" φ and ψ, such as the compactly supported orthogonal ones introduced by Daubechies, are sometimes undesirable for these applications. Conditions on linear-phase ϕ and ψ are established in this paper. In particular, even-order polynomial B-splines and B-wavelets ϕ_m and ψ_m have linear phases, but the odd-order B-wavelet only has generalized linear phases.

1. INTRODUCTION AND NOTATIONS

In this paper, we consider an arbitrary nested sequence

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \tag{1.1}$$

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of closed subspaces that constitute a multiresolution analysis of $L^2 = L^2(\mathbb{R})$ (cf. [14, 16]) and study the family Φ of L^2-functions ϕ such that each $\phi \in \Phi$ generates this multiresolution analysis in the sense that

$$V_k = \operatorname{clos}_{L^2}\langle \phi_{k,j} : j \in \mathbb{Z} \rangle \tag{1.2}$$

for all $k \in \mathbb{Z}$, where

$$\phi_{k,j} = \phi(2^k \cdot - j), \tag{1.3}$$

and that ϕ has a finite two-scale relation, namely

$$\phi(x) = \sum_{n=0}^{N_\phi} p_n^\phi \phi(2x - n), \qquad x \in \mathbb{R}, \tag{1.4}$$

for some finite sequence $\{p_n^\phi\}$. Here and throughout, without loss of generality, we assume that

$$p_0^\phi, p_{N_\phi}^\phi \neq 0, \tag{1.5}$$

so that, as is well known (cf. [9]), ϕ necessarily has compact support, and in fact, we have

$$\operatorname{supp} \phi = [0, N_\phi]. \tag{1.6}$$

The two typical and most important examples of such a ϕ are:

(i) $\phi = N_m$, where N_m is the mth order polynomial B-spline, with knot sequence $\mathbb{Z}$, defined inductively by

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,dt \tag{1.7}$$

with $N_1 = \chi_{[0,1)}$; and

(ii) $\phi = {}_N\phi$ as constructed by Daubechies [9], where $\{{}_N\phi(\cdot - n) : n \in \mathbb{Z}\}$ is an orthonormal family, and the two-scale relation of ${}_N\phi$ yields her compactly supported orthonormal wavelet ${}_N\psi$ by an appropriate shift and an alternation of signs (cf. [9] for the details).

We remark that both N_m and ${}_N\phi$ in the above examples have minimum supports among all of Φ in their corresponding multiresolution analyses. In this paper, we will, however, consider the most general multiresolution analysis (1.1) and any $\phi \in \Phi$ that generates this multiresolution analysis. For convenience, although ϕ is in general not a piecewise polynomial function, we will still call ϕ a (generalized) *spline* and the spaces V_k, $k \in \mathbb{Z}$, *spline spaces*. In particular, if $\varphi \in \Phi$ is the ϕ with minimum support, we will call φ a (generalized) *B-spline*. Our first goal is to establish the basic properties of any $\phi \in \Phi$. This will be done in Section 2. The family Φ will be characterized in Section 3, where the (generalized) B-spline $\varphi \in \Phi$ is identified and an algorithm to determine φ from any $\phi \in \Phi$ is also included.

For each $k \in \mathbb{Z}$, let W_k be the orthogonal complement of V_k in V_{k+1}; that is, $W_k \perp V_k$ and $V_{k+1} = V_k + W_k$, and we will denote this orthogonal sum by

$$V_{k+1} = V_k \oplus W_k. \tag{1.8}$$

A simple consequence of (1.1) and (1.8) is that

$$W_j \perp W_k, \qquad j \neq k, \tag{1.9}$$

and by the properties

$$\operatorname{clos}_{L^2}\left(\bigcup_{k \in \mathbb{Z}} V_k\right) = L^2 \qquad \text{and} \qquad \bigcap_{k \in \mathbb{Z}} V_k = \{0\}, \tag{1.10}$$

of a multiresolution analysis (cf. [14]), we also have

$$L^2 = \bigoplus_{k \in \mathbb{Z}} W_k. \tag{1.11}$$

The orthogonal subspaces W_k, $k \in \mathbb{Z}$, are called *wavelet spaces*, and the orthogonal decomposition (1.11) may be called a *complete wavelet decomposition*. It will be clear that while the spaces V_k, $k \in \mathbb{Z}$, are generated by a single $\phi \in \Phi$ in the sense of (1.2) and (1.3), the wavelet space W_k, $k \in \mathbb{Z}$, are also generated by a single L^2-function η in the same manner, namely

$$W_k = \operatorname{clos}_{L^2}\langle \eta_{k,j} : j \in \mathbb{Z} \rangle, \tag{1.12}$$

where

$$\eta_{k,j} := \eta(2^k \cdot - j). \tag{1.13}$$

If $\eta(x)$ has at least exponential decay as $|x| \to \infty$, then η will be called a *wavelet*, and the collection of all wavelets will be denoted by Ψ. (In applications to time-frequency localization, for example, it is required that both $\eta \in L^1 \cap L^2$ and $x\eta(x) \in L^2$.) We remark, however, that Meyer [17] also considered wavelets with slower decay.

In Section 4, we will discuss the structure of wavelets based on the (generalized) B-spline $\varphi \in \Phi$. In particular, if φ is the polynomial B-spline N_m, we recover the interpolatory spline wavelet in [5] and the compactly supported spline wavelet in [6]. In general, we characterize the minimally supported wavelet ψ in Ψ and call it the *B-wavelet* for this multiresolution analysis and wavelet decomposition. One of the main contributions in Section 4 is the set of formulas for computing the sequences for the two-scale relations and the decomposition relations.

The two sequences that dictate the two-scale relations of $\varphi \in \Phi$ and any wavelet $\eta \in \Psi$ yield a reconstruction (pyramid) algorithm, and the two sequences that define the corresponding decomposition relation give rise to a decomposition (pyramid) algorithm. These are generalizations of the orthonormal setting considered in Mallat [14, 15] and the polynomial spline case in [5]. It will be seen in Section 5 that in order to acquire both finite reconstruction and finite decomposition algorithms it is necessary that both $\{\varphi(\cdot - n): n \in \mathbb{Z}\}$ and $\{\eta(\cdot - n): n \in \mathbb{Z}\}$ are orthonormal families. However, such families, which were constructed in the celebrated paper of Daubechies (cf. [9]), are not symmetric or antisymmetric. Hence, in order to have the desirable linear-phase property, Daubechies' wavelets are not quite desirable. Conditions on the multiresolution that guarantee the properties of generalized linear phase and linear phase for ϕ and η will be established in Section 5. It is noted, in particular, that all even-order polynomial B-splines and B-wavelets have linear phases but the odd-order polynomial B-wavelets only have generalized linear phases.

In general, if we choose $\varphi \in \Phi$ and any compactly supported wavelet $\eta \in \Psi$, then regardless of orthogonality, we always have a finite reconstruction algorithm. In Section 6, we determine the dual bases of $\{\varphi(\cdot - n): n \in \mathbb{Z}\}$ and $\{\eta(\cdot - n): n \in \mathbb{Z}\}$ and show that by interchanging the pair $(\tilde{\varphi}, \tilde{\eta})$ of their duals, we attain a finite decomposition algorithm. That is, the *duality principle* of [6] for polynomial splines also holds for this general setting. The dual bases we introduce here are different from the bi-orthogonal bases considered recently by Cohen [8] and Daubechies [11]. A brief discussion will also be included in this section.

Under Final Remarks, we consolidate the perhaps well-known properties (at least for the orthonormal setting) of approximation order, order of zeros at $2\pi\mathbb{Z}$ of the Fourier transform of the (generalized) B-spline φ, the commutator order of φ, vanishing moments of the B-wavelet ψ, and order of the zero at $z = -1$ of the symbol for the two-scale sequence of φ.

To facilitate our presentation in this paper, we will adopt the following notations and terminologies:

(1°) π denotes the collection of all algebraic polynomials with complex coefficients.

(2°) For any ϕ satisfying (1.4) and (1.5), let P_ϕ denote the polynomial

$$P_\phi(z) = \frac{1}{2} \sum_{n=0}^{N_\phi} p_n^\phi z^n.$$

(3°) $P(\Phi)$ denotes the collection of all polynomials P_ϕ defined in (2°) where $\phi \in \Phi$.

(4°) For each $\phi \in \Phi$, define

$$\gamma_\phi(x) := \int_{-\infty}^{\infty} \phi(x+y)\,\overline{\phi(y)}\,dy.$$

(5°) For $\phi \in \Phi$ and γ_ϕ as defined in (4°), let

$$B_\phi(z) = \sum_{n \in \mathbb{Z}} \gamma_\phi(n)\,z^n.$$

Here, we must pause to remark that from the definition in (4°) and in view of the property (1.6), it is clear that γ_ϕ satisfies

$$\begin{aligned} &\gamma_\phi(-x) = \overline{\gamma_\phi(x)} \\ &\operatorname{supp} \gamma_\phi \subseteq [-N_\phi, N_\phi] \end{aligned} \tag{1.14}$$

so that $B_\phi(z)$ defined in (5°) is a finite Laurent series. We now continue with the above list of notations and terminologies, making simple conclusions in parentheses.

(6°) For $\phi \in \Phi$, let k_ϕ be the non-negative integer such that $\gamma_\phi(k_\phi) \neq 0$ but $\gamma_\phi(n) = 0$ for all $n > k_\phi$. (Hence, from (1.14), we have $\gamma_\phi(-k_\phi) \neq 0$ and $\gamma_\phi(n) = 0$ for all $n < -k_\phi$.)

(7°) Let $\Pi_\phi(z) = z^{k_\phi} B_\phi(z)$. (Hence, Π_ϕ is a polynomial with degree exactly equal to $2k_\phi$.)

(8°) For any $P \in \pi$, let $\check{P}$ denote its reciprocal polynomial. (Hence, we have $\check{\Pi}_\phi(z) = z^{2k_\phi}\,\overline{\Pi_\phi(1/\bar{z})}$.)

(9°) z_0 is called a *symmetric root*, or $(z^2 - z_0^2)$ a *symmetric factor*, of $P \in \pi$, if $z_0 \neq 0$ and $P(z_0) = P(-z_0)$.

(10°) Let $\xi, \tilde{\xi} \in L^2$. We say that ξ and $\tilde{\xi}$ are *duals* to each other if

$$\int_{-\infty}^{\infty} \tilde{\xi}(x-m)\,\overline{\xi(x-n)}\,dx = \delta_{m,n}.$$

(11°) An L^2-function ξ is said to be o.n., if the collection of functions

$$\{\xi(\cdot - n) : n \in \mathbb{Z}\}$$

constitutes an orthonormal family in L^2 in the sense that

$$\int_{-\infty}^{\infty} \xi(x)\,\overline{\xi(x-n)}\,dx = \delta_{n,0}.$$

(Hence, if ξ is o.n., ξ is self-dual, in the sense that $\tilde{\xi} = \xi$.)

(12°) The following normalization of the Fourier transform will be used:

$$\hat{\xi}(\omega) = \int_{-\infty}^{\infty} \xi(x)\, e^{-ix\omega}\, dx.$$

(13°) Let $\Gamma = \{z : z = 0 \text{ or } |z| = 1\}$ and $\Gamma^c = \mathbb{C} \setminus \Gamma$.

(14°) If $|z| = 1$, we will always set

$$z = e^{-i\omega/2}, \tag{1.15}$$

where ω is real.

2. Generators of a Multiresolution Analysis

Let ϕ be a nontrivial function in $L^1 \cap L^2$ and $\phi_{k,j}$ be as difined in (1.3). For each $k \in \mathbb{Z}$, let V_k denote, as in (1.2), the L^2-closure of the linear span of the collection of $\phi_{k,j}$, $j \in \mathbb{Z}$. Following Meyer [16] and Mallat [14, 15], we say that ϕ generates a multiresolution analysis of L^2 if both (1.1) and (1.10) hold and if $\{\phi_{0,j}\}$, $j \in \mathbb{Z}$, is an unconditional basis of V_0.

In what follows, let us restrict our attention on a given fixed multiresolution analysis of nested subspaces (1.1) and consider the collection Φ of all ϕ such that each $\phi \in \Phi$ generates this multiresolution analysis and that ϕ satisfies a two-scale relation governed by a finite sequence. Such a ϕ can certainly be normalized to satisfy

$$\begin{aligned} &\hat{\phi}(0) = 1 \qquad \text{and} \\ &\phi(x) = \sum_{n=0}^{N_\phi} p_n^\phi \phi(2x - n), \qquad p_0^\phi \neq 0,\ p_N^\phi \neq 0. \end{aligned} \tag{2.1}$$

For convenience, we will always assume that each $\phi \in \Phi$ satisfies (2.1) for some positive integer N_ϕ and generates the same multiresolution analysis (1.1). Hence, as is well known (cf. [9]), each $\phi \in \Phi$ satisfies (1.6), so that to look for the $\varphi \in \Phi$ with minimum support, it is equivalent to determining $\varphi \in \Phi$ with

$$N_\varphi = \min\{N_\phi : \phi \in \Phi\}. \tag{2.2}$$

Let us pause for a moment to remark that, at least for the orthonormal setting, sufficient conditions that guarantee the non-emptiness of Φ have been derived by Cohen [7], Daubechies [9, 11], Daubechies and Lagarias [12], and Meyer [16]. For completeness, we will include a brief discussion of this topic under Final Remarks.

Making use of the notations and terminologies introduced in Section 1, we collect a list of basic properties of any generator $\phi \in \Phi$ of the given multiresolution analysis as follows.

THEOREM 2.1. *Every $\phi \in \Phi$ satisfies the following properties.*

(i) *For all $z = e^{-i\omega/2}$, $\omega \in \mathbb{R}$,*

$$\sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 = B_\phi(z^2). \tag{2.3}$$

In particular, ϕ is o.n. in the sense of (11°) *if and only if $B_\phi(z) = 1$ for all $|z| = 1$.*

(ii) $\Pi_\phi(z) \neq 0$ *on* $|z| = 1$.

(iii) $\check{\Pi}_\phi = \Pi_\phi$.

(iv) *For all $|z| = 1$,*

$$|P_\phi(z)|^2 B_\phi(z) + |P_\phi(-z)|^2 B_\phi(-z) = B_\phi(z^2). \tag{2.4}$$

In particular, if ϕ is o.n. in the sense of (11°), *then*

$$|P_\phi(z)|^2 + |P_\phi(-z)|^2 = 1 \tag{2.5}$$

for all $|z| = 1$.

(v) *For all $z \in \mathbb{C}$,*

$$\begin{aligned} P_\phi(z)\, \check{P}_\phi(z)\, \Pi_\phi(z) &+ (-1)^{N_\phi - k_\phi} P_\phi(-z)\, \check{P}_\phi(-z)\, \Pi_\phi(-z) \\ &= z^{N_\phi - k_\phi} \Pi_\phi(z^2). \end{aligned} \tag{2.6}$$

(vi) *P_ϕ has no symmetric roots on $|z| = 1$.*

Remark 2.1. For o.n. ϕ, the condition (2.5) for $|z| = 1$ is important in the study of orthonormal wavelets in [14, 9, 7].

Remark 2.2. The identity (2.6) is instrumental to yielding the wavelet decomposition sequences. For the polynomial spline setting as considered in [5, 6], it reduces to the key identity for the Euler–Frobenius polynomial

$$\begin{aligned} \Pi_{2m-1}(z) &:= (2m-1)!\, \Pi_{N_m}(z) \\ &= (2m-1)! \sum_{j=-m+1}^{m-1} N_{2m}(m+j)\, z^{m+j-1} \end{aligned} \tag{2.7}$$

of degree $2m - 2$ (cf. [19]) as established in [5, Lemma 2]. Recall that the $2m - 2$ roots $r_1, ..., r_{2m-2}$ of Π_{2m-1} are simple, real, negative, and satisfy

$$r_{2m-2} < r_{2m-3} < \cdots < r_m < -1 < r_{m-1} < \cdots < r_1 < 0 \tag{2.8}$$

and

$$r_{2m-2}r_1 = r_{2m-3}r_2 = \cdots = r_m r_{m-1} = 1. \tag{2.9}$$

Hence, it is clear that Π_{2m-1} satisfies (ii) and (iii) in Theorem 2.1.

Proof. (i) Let $z = e^{-i\omega/2}$ be fixed and consider $g(x) := \gamma_\phi(x)\, e^{-ix\omega}$ (cf. (4°) for the definition of γ_ϕ). Then

$$\begin{aligned}\hat{g}(s) &= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} \phi(x+y)\,\overline{\phi(y)}\,dy\right\} e^{-ix\omega}\, e^{-ixs}\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x+y)\, e^{-i(x+y)(\omega+s)}\,\overline{\phi(y)\, e^{-iy(\omega+s)}}\,dx\,dy \\ &= \hat{\phi}(\omega+s)\,\overline{\hat{\phi}(\omega+s)} = |\hat{\phi}(\omega+s)|^2.\end{aligned}$$

Hence, by the Poisson Summation formula, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+2\pi n)|^2 &= \sum_{n\in\mathbb{Z}} \hat{g}(2\pi n) \\ &= \sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \gamma_\phi(n)\, e^{-in\omega} \\ &= \sum_{n\in\mathbb{Z}} \gamma_\phi(n)\, z^{2n} = B_\phi(z^2)\end{aligned}$$

(cf. (5°)). In particular, for o.n. ϕ, we have $\gamma_\phi(n) = \delta_{n,0}$ so that $B_\phi \equiv 1$.

(ii) Since $\{\phi_{0,n} : n\in\mathbb{Z}\}$ is an unconditional basis, there exist constants A and B, with $0 < A \leqslant B < \infty$, such that

$$A\,\|\mathbf{c}\|_{l^2}^2 \leqslant \left\|\sum_{n\in\mathbb{Z}} c_n \phi_{0,n}\right\|_{L^2}^2 \leqslant B\,\|\mathbf{c}\|_{l^2}^2 \tag{2.10}$$

for all l^2 sequences $\mathbf{c} = \{c_n\}$. Set

$$C(\omega) := \sum_{n\in\mathbb{Z}} c_n\, e^{-in\omega}.$$

Then by the 2π-periodicity of $C(\omega)$, we have

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}} c_n \phi_{0,n}\right\|_{L^2}^2 &= \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}} c_n \hat{\phi}_{0,n}\right\|_{L^2}^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |C(\omega)\,\hat{\phi}(\omega)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} |C(\omega)|^2 \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+2\pi n)|^2\,d\omega.\end{aligned}$$

Hence, it follows from (2.10) and the identity

$$\frac{1}{2\pi}\int_0^{2\pi} |C(\omega)|^2\, d\omega = \|\mathbf{c}\|_{l^2}^2$$

that

$$A \leqslant \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+2\pi n)|^2 \leqslant B \tag{2.11}$$

for all $\omega\in\mathbb{R}$. In particular, by (i) we have

$$|\Pi_\phi(z)| = |B_\phi(z)| \geqslant A > 0$$

on $|z|=1$ (cf. (7°)).

(iii) Since $\gamma_\phi(-n)=\overline{\gamma_\phi(n)}$ by (1.14), we have, for $|z|=1$,

$$\begin{aligned}\check{\Pi}_\phi(z) &:= z^{2k_\phi}\sum_{n=-k_\phi}^{k_\phi} \overline{\gamma_\phi(n)}\, z^{-n-k_\phi}\\ &= z^{k_k}\sum_{n=-k_\phi}^{k_\phi} \gamma_\phi(-n)\, z^{-n}\\ &= z^{k_\phi}\sum_{n=-k_\phi}^{k_\phi} \gamma_\phi(n)\, z^{n} = \Pi_\phi(z).\end{aligned}$$

But since both $\check{\Pi}_\phi$ and Π_ϕ are polynomials, they must be identical.

(iv) From the definition of P_ϕ in (2°), taking the Fourier transform of both sides of the two-scaled formula (2.1) yields

$$\hat{\phi}(\omega) = P_\phi(z)\hat{\phi}\left(\frac{\omega}{2}\right), \qquad z=e^{-i(\omega/2)}. \tag{2.12}$$

Hence, for $z=e^{-i\omega/2}$, (2.3) becomes

$$\begin{aligned}B_\phi(z^2) &= \sum_{n\in\mathbb{Z}} \left|P_\phi((-1)^n z)\,\hat{\phi}\left(\frac{\omega}{2}+\pi n\right)\right|^2\\ &= |P_\phi(z)|^2 \sum_{n\in\mathbb{Z}}\left|\hat{\phi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2\\ &\quad + |P_\phi(-z)|^2\sum_{n\in\mathbb{Z}}\left|\hat{\phi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2\\ &= |P_\phi(z)|^2\, B_\phi(z) + |P_\phi(-z)|^2\, B_\phi(-z)\end{aligned}$$

which is (2.4). Of course, if ϕ is o.n., then (2.4) becomes (2.5) since B_ϕ is the constant 1.

(v) By appealing to the formulas

$$\overline{P_\phi(z)} = z^{-N_\phi}\check{P}_\phi(z) \qquad \text{and} \qquad B_\phi(z) = z^{-k_\phi}\Pi_\phi(z),$$

which hold for $|z| = 1$, it is easy to see that (2.4) is equivalent to (2.6) for all z, $|z| = 1$. But since both sides of (2.6) are polynomials, the formula (2.6) holds for all $z \in \mathbb{C}$.

(vi) If $z_0 \neq 0$ is a symmetric root of P_ϕ, then we have $P_\phi(z_0) = P_\phi(-z_0) = 0$, so that $\Pi_\phi(z_0^2) = 0$ by (2.6) in (v). Hence, by (ii), we conclude that $|z_0^2| \neq 1$; that is, P_ϕ has no symmetric roots on $|z| = 1$. ∎

Remark 2.3. In the proof of (ii), we have actually established the equivalence of (2.10) and (2.11). Hence, if $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is a basis of V_0, then this basis is unconditional, provided that (2.11) is satisfied for all ω and some positive constants A and B.

3. Minimally Supported Generators and Characterization of Φ

The first goal in this section is to characterize the minimally supported $\varphi \in \Phi$ in terms of the structure of its associated (two-scale) polynomial P_φ. It will be seen that φ is unique and, for convenience, we call it the (*generalized*) *B-spline* of this given multiresolution analysis. By using φ, we can characterize the whole class Φ, and from this characterization it will be clear how φ can be obtained from any $\phi \in \Phi$.

Recall from (2.1), (1.6), (2.2), and the definition of the polynomial P_ϕ in (2°), that $\varphi \in \Phi$ has minimum support among all $\phi \in \Phi$ if and only if

$$\deg P_\varphi = \min\{\deg P_\phi : \phi \in \Phi\}. \tag{3.1}$$

We have the following result.

THEOREM 3.1. *$\varphi \in \Phi$ has minimum support if and only if P_φ has no symmetric roots.*

Proof. (i) Suppose that $\varphi \in \Phi$ has minimum support and consider the factorization,

$$P_\varphi(z) = m_\varphi(z)\, n_\varphi(z^2),$$

where $m_\varphi, n_\varphi \in \pi$, m_φ has no symmetric roots, and $m_\varphi(1) = n_\varphi(1) = 1$ (cf. (2.12) with $\omega = 0$). We must prove that n_φ is the constant 1. Assume, on the contrary, that this is not true. Then $\deg n_\varphi \geqslant 1$. Since P_φ has no symmetric roots on $|z| = 1$ (cf. Theorem 1.1(vi)), the polynomial n_φ cannot vanish on $|z|=1$, so that n_φ^{-1} is analytic in a neighborhood of the unit circle. Consider the series

$$\frac{1}{n_\varphi(z)}=\sum_{n\in\mathbb{Z}} r_n z^n, \qquad |z|=1, \tag{3.2}$$

where $\{r_n\}\in l^2$, and define

$$\phi(x)=\sum_{n\in\mathbb{Z}} r_n\varphi(x-n).$$

Then by using the notation $z=e^{-i\omega/2}$ and applying (2.12), we have

$$\begin{aligned}
\hat{\phi}(\omega)&=\left(\sum_{n\in\mathbb{Z}} r_n e^{-in\omega}\right)\hat{\varphi}(\omega)\\
&=\frac{1}{n_\varphi(z^2)}P_\varphi(z)\,\hat{\varphi}\left(\frac{\omega}{2}\right)\\
&=m_\varphi(z)\,\hat{\varphi}\left(\frac{\omega}{2}\right)\\
&=m_\varphi(z)\,\hat{\phi}\left(\frac{\omega}{2}\right)\left(\sum_{n\in\mathbb{Z}} r_n e^{-in\omega/2}\right)^{-1}\\
&=(m_\varphi(z)\,n_\varphi(z))\,\hat{\phi}\left(\frac{\omega}{2}\right).
\end{aligned} \tag{3.3}$$

That is, $\phi\in\Phi$ with

$$P_\phi(z)=m_\varphi(z)\,n_\varphi(z)$$

so that

$$\deg P_\phi(z)<\deg(m_\varphi(z)\,n_\varphi(z^2))=\deg P_\varphi(z).$$

This contradicts (3.1).

(ii) Conversely, suppose that P_φ has no symmetric roots. Let $\phi\in\Phi$ be arbitrarily chosen and write

$$\phi(x)=\sum_{n\in\mathbb{Z}} c_n\varphi(x-n),$$

where $\{c_n\}\in l^2$. Then

$$\hat{\phi}(\omega)=C(z^2)\,\hat{\varphi}(\omega), \qquad z=e^{-i\omega/2}, \tag{3.4}$$

with

$$C(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

Hence, by the 2π-periodicity of $C(z^2)$, we have

$$\sum_{n \in \mathbb{Z}} \hat{\phi}(\omega + 2\pi n) = C(z^2) \sum_{n \in \mathbb{Z}} \hat{\varphi}(\omega + 2\pi n). \tag{3.5}$$

In view of the Poisson Summation formula, it is clear that (3.5) is equivalent to

$$\sum_{n \in \mathbb{Z}} \phi(n)\, e^{-in\omega} = C(z^2) \sum_{n \in \mathbb{Z}} \varphi(n)\, e^{-in\omega}, \tag{3.6}$$

$z = e^{-i\omega/2}$. That is, from (1.6), we see that $C(z)$ is a rational function. In addition, again by the 2π-periodicity of $C(z^2)$, it follows from Theorem 2.1(i) and (3.4) that

$$|C(z^2)|^2 = B_\phi(z^2)/B_\varphi(z^2),$$

where $|B_\phi(z)| = |\Pi_\phi(z)|$ and $|B_\varphi(z)| = |\Pi_\varphi(z)|$ for $|z| = 1$ by (7°). Since both of the polynomials Π_ϕ and Π_φ do not vanish on $|z| = 1$ (cf. Theorem 2.1(ii)), we may conclude that the rational function $C(z)$ is pole-free and zero-free on $|z| = 1$.

Next, consider the factorizations

$$\sum_{n \in \mathbb{Z}} \phi(n)\, z^n = \sum_{n=0}^{N_\phi} \phi(z)\, z^n = \xi_\phi(z)\, d(z) \tag{3.7}$$

and

$$\sum_{n \in \mathbb{Z}} \varphi(n)\, z^n = \sum_{n=0}^{N_\varphi} \varphi(n)\, z^n = \xi_\varphi(z)\, d(z) \tag{3.8}$$

(cf. (1.6)), where the two polynomials ξ_ϕ and ξ_φ have no common factors and $\xi_\phi(0)$, $\xi_\varphi(0) \neq 0$. Then it follows from the two-scale equations of ϕ and φ and (3.4), that for $z = e^{-i\omega/2}$,

$$\begin{aligned}
\hat{\phi}(\omega) &= P_\phi(z)\, \hat{\phi}\left(\frac{\omega}{2}\right) = C(z)\, P_\phi(z)\, \hat{\varphi}\left(\frac{\omega}{2}\right) \\
&= C(z)\, P_\phi(z)\, \hat{\varphi}(\omega)/P_\varphi(z) \\
&= \frac{C(z)}{C(z^2)}\, P_\phi(z)\, \hat{\varphi}(\omega)/P_\varphi(z),
\end{aligned}$$

so that, in view of (3.6)–(3.8), we have

$$P_\phi(z) = \frac{C(z^2)}{C(z)} P_\varphi(z) = \frac{\xi_\varphi(z^2)\,\xi_\varphi(z)}{\xi_\phi(z)\,\xi_\varphi(z^2)} P_\varphi(z). \tag{3.9}$$

Now since P_ϕ is a polynomial and the two polynomials $\xi_\phi(z^2)$ and $\xi_\varphi(z^2)$ do not have any common factors, we must have

$$\xi_\varphi(z)\,P_\varphi(z) = \zeta(z)\,\xi_\varphi(z^2) \tag{3.10}$$

for some polynomial ζ. Now, by (3.6)–(3.8), we have $\xi_\phi(z) = C(z)\,\xi_\varphi(z)$. So, since ξ_ϕ and ξ_φ have no common factors and the rational function $C(z)$ is pole-free and zero-free on $|z| = 1$, it follows that ξ_φ is zero-free on $|z| = 1$ also. From the normalization $\xi_\varphi(0) \neq 0$, it is therefore clear that if ξ_φ is not a constant, and we write

$$\xi_\varphi(z) = \xi_\varphi(0) \prod_{j=1}^{p} \left(1 - \frac{z}{z_j}\right)$$

for some $p \geqslant 1$ and

$$\begin{aligned}\xi_\varphi(z^2) &= \xi_\varphi(0) \prod_{j=1}^{p} \left(1 - \frac{z^2}{z_j}\right) \\ &= \xi_\varphi(0) \left[\prod_{j=1}^{p} \left(1 - \frac{z}{z_j'}\right)\right]\left[\prod_{j=1}^{p} \left(1 + \frac{z}{z_j'}\right)\right],\end{aligned}$$

where z_j' is a branch of the square-root of z_j, then the two point-sets $\{z_j : j = 1, ..., p\}$ and $\{z_j' : j = 1, ..., p\}$ do not coincide. Hence, from (3.10) there is some z_{j_0}, $1 \leqslant j_0 \leqslant p$, such that $(z^2 - z_{j_0})$ is a factor of P_φ, contradicting that P_φ has no symmetric roots; unless of course, that ξ_φ is a constant. Hence, it follows from (3.9) that

$$P_\phi(z) = \frac{\xi_\phi(z^2)}{\xi_\phi(z)} P_\varphi(z),$$

so that $\deg P_\phi = \deg \xi_\phi + \deg P_\varphi \geqslant \deg P_\varphi$. In other words, φ has minimum support. This completes the proof of Theorem 3.1. ∎

As a consequence of Theorem 3.1, we have the following uniqueness result.

COROLLARY 3.1. *Let $\varphi \in \Phi$ have minimum support. Then*

(i) *every $\phi \in \Phi$ is a finite linear combination of integer translates of φ; and*

(ii) *φ is unique.*

Proof. (i) From the proof of the above theorem and using (3.6)–(3.8), since ξ_φ is a constant, we have $C(z) = (1/\xi_\varphi(0))\, \xi_\phi(z) \in \pi$. Hence, the statement (i) follows from (3.4).

(ii) Let $\tilde{\varphi} \in \Phi$ also have minimum support. By (i), we may write

$$\tilde{\varphi}(x) = \sum_{j=0}^{p} c_j\, \varphi(x-j), \qquad c_p \neq 0,$$

for some finite integer $p \geqslant 0$. Here, let us recall that

$$\operatorname{supp} \tilde{\varphi} = [0, N_{\tilde{\varphi}}]$$
$$\operatorname{supp} \varphi(\cdot - j) = [j, N_\varphi + j], \qquad j = 0, ..., p.$$

Since $\tilde{\varphi}$ has minimum support, we have $N_{\tilde{\varphi}} = N_\varphi$, so that, if $p \geqslant 1$, then for $x \in [N_\varphi + p - 1, N_\varphi + p]$,

$$0 = \tilde{\varphi}(x) = \sum_{j=0}^{p} c_j\, \varphi(x-j) = c_p \varphi(x-p).$$

But that $\varphi(x)$ is nontrivial on $[N_\varphi - 1, N_\varphi]$ implies that $\varphi(x-p)$ is nontrivial on $[N_\varphi + p - 1, N_\varphi + p]$, and this, in turn, implies that $c_p = 0$, a contradiction. That is, $\tilde{\varphi} = c_0 \varphi$. Hence, using the normalization $\hat{\tilde{\varphi}}(0) = \hat{\varphi}(0) = 1$, we have $\tilde{\varphi} = \varphi$. ∎

As another consequence of Theorem 3.1, we also have the following result.

COROLLARY 3.2. *If there is a* $\phi \in \Phi$ *which is o.n. in the sense of* (11^c), *then* ϕ *has minimum support.*

Proof. If $\phi \in \Phi$ is o.n., then by (2.6) in Theorem 2.1(v), we have

$$P_\phi(z)\, \check{P}_\phi(z) + (-1)^{N_\phi} P_\phi(-z)\, \check{P}_\phi(-z) = z^{N_\phi}, \qquad z \in \mathbb{C}.$$

So, P_ϕ cannot have any symmetric root, and it follows from Theorem 3.1 that ϕ has minimum support. ∎

From the (unique) minimally supported $\varphi \in \Phi$, we will now characterize all of Φ. First, let us recall the notations of Γ and Γ^c in (13°). For any $P \in \pi$, we introduce the two classes of polynomials:

$$L(P) = \left\{ l \in \pi : l(z) \neq 0 \text{ for } z \in \Gamma,\ l(1) = 1,\ \frac{l(z^2)}{l(z)} P(z) \in \pi \right\} \qquad (3.11)$$

and

$$R(P)=\left\{r\in\pi: r(z)\neq 0 \text{ for } z\in\Gamma,\, r(1)=1,\, \frac{r(z)}{r(z^2)}P(z)\in\pi\right\}. \tag{3.12}$$

Remark 3.1. If $P(z)\neq 0$ for all $z\in\Gamma^c$, then

$$L(P)=R(P)=\{1\}.$$

EXAMPLE 3.1. Consider the polynomial spline spaces with $\phi=N_m$ denoting the mth order polynomial B-spline defined in (1.7). Then it is well known that

$$P_{N_m}(z)=\left(\frac{1+z}{2}\right)^m$$

(cf. [4]), so that $L(P_{N_m})=R(P_{N_m})=\{1\}$.

Also recall from (3°) that $P(\Phi)$ denotes the class of all polynomials P_ϕ with $\phi\in\Phi$. Hence, characterizing Φ is equivalent to identifying all the polynomials in $P(\Phi)$. We have the following result.

THEOREM 3.2. *Let $\varphi\in\Phi$ have minimum support. Then*

$$P(\Phi)=\left\{p\in\pi: p(z)=\frac{l(z^2)}{l(z)}P_\varphi(z),\, l\in L(P_\varphi)\right\}.$$

In particular, if $\phi\in\Phi$, then there exists an $r\in R(P_\phi)$ such that

$$\frac{r(z)}{r(z^2)}P_\phi(z)=P_\varphi(z). \tag{3.13}$$

Remark 3.2. Before we establish this result, let us first remark that in all our discussions in this paper, we allow complex-valued functions. If only real-valued functions $\phi\in\Phi$ are considered, then we simply restrict $l\in L(P_\varphi)$ and $r\in R(P_\phi)$ to those with only real coefficients for the above theorem to be valid.

We now turn to the proof of Theorem 3.2.

Proof. (i) Let $P_\phi\in P(\Phi)$; that is, $\phi\in\Phi$. Then by Corollary 3.1(i), we may write

$$\phi(x)=\sum_{j=0}^{p} c_j\,\varphi(x-j), \qquad c_p\neq 0.$$

Since supp $\phi = [0, N_\phi]$ and supp $\varphi[0, N_\varphi]$, it is also clear that $c_0 \neq 0$. Set

$$l(z) = \sum_{n=0}^{p} c_n z^n,$$

where $l(0) = c_0 \neq 0$. Also, as in the proof of Theorem 3.1, we note that

$$|l(z)|^2 = \frac{|\Pi_\phi(z)|}{|\Pi_\varphi(z)|} \neq 0 \qquad \text{for all} \quad |z| = 1.$$

Hence, $l(z) \neq 0$ for all $z \in \Gamma$. In addition, from the relations

$$\hat{\phi}(\omega) = l(z^2)\, \hat{\varphi}(\omega)$$

$$\hat{\phi}(\omega) = P_\phi(z)\, \hat{\phi}\left(\frac{\omega}{2}\right)$$

$$\hat{\varphi}(\omega) = P_\varphi(z)\, \hat{\varphi}\left(\frac{\omega}{2}\right)$$

which hold for all $z = e^{-i\omega/2}$, we immediately have

$$l(1) = 1$$

$$\frac{l(z)}{l(z^2)} P_\phi(z) = P_\varphi(z), \qquad |z| = 1.$$

By analytical continuation, we now conclude that

$$P_\phi(z) = \frac{l(z^2)}{l(z)} P_\varphi(z), \qquad \text{all } z,$$

so that $l \in L(P_\varphi)$.

(ii) On the other hand, suppose that

$$p(z) := \frac{l(z^2)}{l(z)} P_\varphi(z) \in \pi$$

for some $l \in L(P_\varphi)$. Then for

$$l(z) = \sum_{j=0}^{m} c_j z^j, \qquad c_m \neq 0,$$

we define

$$\phi(x) := \sum_{j=0}^{m} c_j \varphi_j (x - j),$$

so that $P_\phi(z)=p(z)$. Since $l(z)\neq 0$ for all $z\in\Gamma$, it is clear that $\phi\in\Phi$ with $p_0^\phi=P_\varphi(0)=p_0^\varphi\neq 0$ and $p_{N_\phi}^\phi=p_{N_\varphi}^\varphi\neq 0$, where $N_\phi=m+N_\varphi$.

(iii) If $\phi\in\Phi$, then as in the derivation in (i), it is easy to see that by setting $r(r)=l(z)$ in the derivation, we have $r\in R(P_\phi)$ and (3.13) holds.

This completes the proof of the theorem. ∎

The following consequence is useful in the study of generators of the given multiresolution analysis.

COROLLARY 3.3. *Let $\varphi\in\Phi$ have minimum support. If all the roots of P_φ lie on Γ, then $\#\Phi=1$. In other words, φ is the only generator of the given multiresolution analysis which is governed by a finite two-scale sequence.*

Proof. Let $l\in L(P_\varphi)$. Then since $l(z)\neq 0$ on Γ, by the same argument as in the proof of the converse in Theorem 3.1, it is clear that $l(z)$ does not divide $l(z^2)$ unless $l(z)=1$. So, since l and P_φ have no common roots, the function

$$p(z)=\frac{l(z^2)}{l(z)}P_\varphi(z)$$

cannot be a polynomial unless $l(z)$ is the constant function, or equivalently, $\#\Phi=1$. ∎

EXAMPLE 3.2. For the polynomial spline spaces, since the mth order B-spline N_m has the property that $P_{N_m}(z)=0$ only at $z=-1$, it is the *only* multiresolution analysis generator that has a finite two-scale sequence.

EXAMPLE 3.3. For any integer $m\geqslant 1$, consider

$$P_\phi(z):=\left(\frac{1+z}{2}\right)^m M_m(z), \tag{3.14}$$

where $M_m\in\pi$ with $M_m(1)=1$ and

$$\max_{|z|=1}|M_m(z)|<2^{m-1}. \tag{3.15}$$

Then it can be shown that the function ϕ with the finite two-scale sequence defined by (3.14) generates a multiresolution analysis of L^2 (cf. under Final Remarks). Hence, there is a lot of freedom to choose M_m, so that $\#\Phi=\infty$. For instance, for $m=3$, we may set

$$M_3(z)=2(z^{2^k}-\tfrac{1}{2})$$

for any $k=0, 1, 2, \ldots$.

Remark 3.3. The result (3.13) in Theorem 3.2 suggests an algorithm to produce the minimally supported $\varphi \in \Phi$ from any $\phi \in \Phi$. Indeed, if we set

$$P_\phi(z) = m_\phi(z)\, r_0(z^2), \tag{3.16}$$

where m_ϕ and $r_0 \in \pi$, $r_0(1) = 1$, and m_ϕ has no symmetric roots, then we can compute φ as follows.

If $\deg r_0 = 0$, then set $\varphi = \phi$. Otherwise, consider

$$P_{\phi,1}(z) = m_\phi(z)\, r_0(z)$$

and set $P_{\phi,1}(z) = m_{\phi,1}(z)\, r_1(z^2)$, where $m_{\phi,1}, r_1 \in \pi$, $r_1(1) = 1$, and $m_{\phi,1}$ has no symmetric roots; and set $P_{\phi,2}(z) = m_{\phi,1}(z)\, r_1(z)$, etc. Since

$$\deg P_\phi > \deg P_{\phi,1} > \deg P_{\phi,2} > \cdots,$$

this process must terminate at

$$P_{\phi,l}(z) = m_{\phi,l}(z),$$

say, where r_l is the constant 1. Then φ is determined by $P_\varphi = P_{\phi,l}$.

4. Wavelets

While the nested sequence of closed subspaces V_k of the given multiresolution analysis of L^2 is important for approximation purposes, the orthogonal complementary wavelet subspaces W_k, $k \in \mathbb{Z}$, defined in (1.8), are essential for analyzing the behavior of the approximants from V_k. Let $\varphi \in \Phi$ be the (generalized) *B*-spline that generates this multiresolution analysis. Then since we have $W_0 \subset V_1$, any η that generates the wavelet spaces W_k in the sense of (1.12)–(1.13) must satisfy

$$\eta(x) = \sum_{n \in \mathbb{Z}} q_n^\eta \varphi(2x - n), \qquad \{q_n^\eta\} \in l^2. \tag{4.1}$$

For practical purposes such as applications to time-frequency analysis, however, η must be more restrictive. We will require η to have at least exponential decay as follows.

Definition 4.1. An L^2-function η is called a *wavelet* for the given multiresolution analysis provided that

(i) $\{\eta(\cdot - n) : n \in \mathbb{Z}\}$ is an unconditional basis of W_0, and

(ii) the "symbol" function

$$Q_\eta(z) := \tfrac{1}{2} \sum_{n \in \mathbb{Z}} q_n^\eta z^n$$

of the "two-scale sequence" $\{q_n^\eta\}$ in (4.1) is analytic in a neighborhood of $|z|=1$.

The collection of all wavelets η will be denoted by Ψ.

Remark 4.1. The analyticity condition of Q_η on $|z|=1$ is equivalent to the condition that both sequences $\{q_n^\eta\}$ and $\{q_{-n}^\eta\}$ have exponential decay as $n \to +\infty$. Hence, since $\varphi \in \Phi$ has compact support, $\eta \in \Psi$ must have exponential decay.

Remark 4.2. For each $\eta \in \Psi$, by defining $\eta_{k,j} = \eta(2^k \cdot - j)$, it is clear that for each $k \in \mathbb{Z}$, $\{\eta_{k,j} : j \in \mathbb{Z}\}$ is also an unconditional basis of W_k. Hence, every $\eta \in \Psi$ generates all the wavelet spaes W_k, $k \in \mathbb{Z}$.

One of the objectives of this section is to give a characterization of all wavelets $\eta \in \Psi$. The polynomial

$$\beta_\varphi(z) := z^{N_\varphi - k_\varphi - 1} \Pi_\varphi(z)\, \check{P}_\varphi(z) \tag{4.2}$$

will be crucial for this purpose. (Recall the notations of N_φ, k_φ, Π_φ, and $\check{P}_\varphi$ from (2°) and (6°)–(8°) in Section 1.) Let us factorize β_φ as

$$\beta_\varphi(z) = \mu_\varphi(z)\, \lambda_\varphi(z^2), \tag{4.3}$$

where μ_φ, $\lambda_\varphi \in \pi$ such that $\lambda_\varphi(1) = 1$, z^2 does not divide $\mu_\varphi(z)$, and μ_φ has no symmetric roots.

Remark 4.3. Since Π_ϕ is zero-free on $|z|=1$ and P_ϕ has no symmetric roots on $|z|=1$, λ_φ is zero-free on $|z|=1$.

In stating our characterization theorem, we need the following notation.

DEFINITION 4.2. Let $\mathscr{A}$ denote the class of all functions f analytic in a neighborhood of $|z|=1$ such that $f(z) \neq 0$ on $|z|=1$.

The following observation is trivial.

Remark 4.4. $f \in \mathscr{A}$ if and only if $1/f \in \mathscr{A}$. Furthermore, $fg \in \mathscr{A}$ if both f and g are in $\mathscr{A}$. The main result in this section is the following characterization theorem.

THEOREM 4.1. *Let μ_φ be as defined in* (4.3). *Then $\eta \in \Psi$ if and only if*

$$Q_\eta(z) = \mu_\varphi(-z)\, w_\eta(z^2), \tag{4.4}$$

where $w_\eta \in \mathscr{A}$.

In view of the characterization (4.4), it is clear that the rate of decay of the wavelet $\eta \in \Psi$ is governed by the function $w_\eta \in \mathscr{A}$. In fact, we have the following.

COROLLARY 4.1. *Let* μ_φ *be as defined in* (4.3) *and* $w_\eta \in \pi$. *Then the wavelet* η *has compact support.*

We note that Auscher [1] considered $Q_\eta(z) = z^{\varepsilon_\varphi} \check{P}_\varphi(-z)\, \Pi_\varphi(-z)$ and obtained his $\hat{\eta}(\omega)$. This means that he chose $w_\eta(z^2)\, z^{-(N_\varphi - k_\varphi - 1) + \varepsilon_\varphi}$.

Remark 4.5. The condition $w_\eta \in \pi$ is not only a guarantee that the corresponding wavelet η has compact support, it is also a necessary and sufficient condition for the sequence $\{q_n^\eta\}$ in the two-scale formula (4.1) to be a finite sequence. Since the pair of two-scale sequences $\{p_n^\varphi\}$ and $\{q_n^\eta\}$ are used to describe the reconstruction algorithm (a topic to be discussed in the next section; see also [15, 9]), it is very important to restrict w_η to π. For the purpose of normalization, since it may not be possible to assume $q_0^\eta \neq 0$, we will require $q_1^\eta \neq 0$ whenever $q_0^\eta = 0$.

DEFINITION 4.3.

$$\Psi_p = \{\eta \in \Psi \colon Q_\eta \in \pi \text{ and } z^2 \nmid Q_\eta(z)\}. \tag{4.5}$$

Of course for $\eta \in \Psi_p$, the polynomial factor w_η in (4.5) must be zero-free on $|z| = 1$. It is clear that the finite sequence $\{q_m^\eta\}$ is the shortest, or equivalently, the support of η is the smallest (cf. [9]), if and only if w_η is a constant polynomial. That is, we have the following result.

COROLLARY 4.2. $\psi \in \Psi_p$ *has minimum support if and only if* $w_\psi(z) = 1$, *or equivalently*

$$Q_\psi(z) = \mu_\varphi(-z). \tag{4.6}$$

Furthermore, under the normalization $\lambda_\varphi(1) = 1$ *in* (4.3), *the minimally supported wavelet* ψ *is unique.*

DEFINITION 4.4. The minimally supported wavelet $\psi \in \Psi_p$ is called the *B-wavelet* of the given multiresolution analysis.

COROLLARY 4.3. *If the minimally supported* $\varphi \in \Phi$ *is o.n. in the sense of* ($11°$), *then the* (*minimally supported*) *B-wavelet* $\psi \in \Psi_p$ *is also o.n. in the sense of* ($11°$) *and is given by*

$$\psi(x) = \sum_{n = \varepsilon_\varphi}^{N_\varphi + \varepsilon_\varphi} (-1)^n\, \overline{p^\varphi_{N_\varphi - n + \varepsilon_\varphi}}\, \varphi(2x - n), \tag{4.7}$$

where $\varepsilon_\varphi := 1$ *for even* N_φ *and* $\varepsilon_\varphi := 0$ *for odd* N_φ.

Remark 4.6. The o.n. wavelet ψ in Daubechies [9] is shifted by $(N_\varphi + \varepsilon_\varphi - 1)/2$.

EXAMPLE 4.1. Consider the polynomial spline setting where φ is the mth order B-spline N_m. Then in the factorization (4.3), we have $\lambda_{N_m}(z)=1$, so that $\mu_{N_m}=\beta_{N_m}$ as defined in (4.2). In fact, we have

$$\mu_{N_m}(z)=\frac{1}{(2m-1)!}\left(\frac{1+z}{2}\right)^m \Pi_{2m-1}(z),$$

where Π_{2m-1} is the Euler–Frobenius polynomial of degree $2m-2$. Hence, by choosing

$$w_{\eta_m}(z^2):=\frac{2^m((2m-1)!)^2}{\Pi_{2m-1}(z)\,\Pi_{2m-1}(-z)}$$

in (4.4) we have

$$Q_{\eta_m}(z)=(2m-1)!\,\frac{(1-z)^m}{\Pi_{2m-1}(z)},$$

which gives the wavelet

$$\eta_m(x)=L_{2m}^{(m)}(2x-1) \tag{4.8}$$

obtained in [5], where L_{2m} is the $(2m)$th order "fundamental spline" with knots at $\mathbb{Z}$, defined by the interpolation condition $L_{2m}(n)=\delta_{n,0}$ for all $n\in\mathbb{Z}$. In addition, by choosing $\omega_{\psi_m}(z)=1$, we have

$$Q_{\psi_m}(z)=\frac{1}{(2m-1)!}\left(\frac{1-z}{2}\right)^m \Pi_{2m-1}(-z)$$

which yields the compactly supported wavelet

$$\psi_m(x)=\frac{1}{2^{m-1}}\sum_{j=0}^{2m-2}(-1)^j N_{2m}(j+1)\,N_{2m}^{(m)}(2x-j) \tag{4.9}$$

obtained in [6]. Note that $\operatorname{supp}\psi_m=[0,2m-1]$, and by Corollary 4.2, this is the (minimally supported) *B-wavelet* of the polynomial spline spaces.

We now turn to the proof of Theorem 4.1.

Proof. (i) Suppose that w_η is in the class $\mathscr{A}$ and Q_η is defined as in (4.4). First, we observe that since $\mu_\varphi\in\pi$, Q_η is analytic in a neighborhood of $|z|=1$.

Secondly, let us verify that $\eta(\cdot-n)$ is in W_0 for all n. We need to reformulate (4.1) as

$$\hat{\eta}(\omega)=Q_\eta(z)\,\hat{\varphi}\left(\frac{\omega}{2}\right),\qquad z=e^{-i\omega/2}. \tag{4.10}$$

Then for all $l_1, l_2 \in \mathbb{Z}$, setting $k = l_2 - l_1$, we have, again using the notation $z = e^{-i\omega/2}$,

$$\begin{aligned}
I_k &:= \int_{-\infty}^{\infty} \varphi(x-l_1)\,\overline{\eta(x-l_2)}\,dx = \int_{-\infty}^{\infty} \varphi(x)\,\overline{\eta(x-k)}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)\,\overline{\hat{\eta}(\omega)}\,e^{ik\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\omega} P_\varphi(z)\,\overline{Q_\eta(z)}\left|\hat{\varphi}\left(\frac{\omega}{2}\right)\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{0}^{4\pi} e^{ik\omega} P_\varphi(z)\,\overline{Q_\eta(z)} \sum_{n\in\mathbb{Z}}\left|\hat{\varphi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{ik\omega}[P_\varphi(z)\,\overline{Q_\eta(z)}\,B_\varphi(z) + P_\varphi(-z)\,\overline{Q_\eta(-z)}\,B_\varphi(-z)]\,d\omega,
\end{aligned}$$

where the last equality is a consequence of Theorem 2.1(i). Now, from (4.2), (4.3), (4.5), and (7°), it is clear that

$$\begin{aligned}
Q_\eta(z) &= \mu_\varphi(-z)\,w_\eta(z^2) = \frac{\beta_\varphi(-z)\,w_\eta(z^2)}{\lambda_\varphi(z^2)} \\
&= \frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}[(-z)^{N_\varphi-1}B_\varphi(-z)\,\check{P}_\varphi(-z)].
\end{aligned}$$

Also, recalling from (5°), (1.14), and the definition of $\check{P}_\varphi$, we have

$$\begin{aligned}
\overline{B(z)} &= B(z) \\
\check{P}_\varphi(z) &= z^{N_\varphi}\overline{P(z)}
\end{aligned}$$

for $|z| = 1$. Hence, it follows that, for $|z| = 1$,

$$\begin{aligned}
&P_\varphi(z)\,\overline{Q_\eta(z)}\,B_\varphi(z) + P_\varphi(-z)\,\overline{Q_\eta(-z)}\,B_\varphi(-z) \\
&\quad = \overline{\left[\frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}\right]}\,B_\varphi(z)\,B_\varphi(-z) \\
&\qquad \times [(-z)^{-N_\varphi+1}P_\varphi(z)\,\overline{\check{P}_\varphi(-z)} + z^{-N_\varphi+1}P_\varphi(-z)\,\overline{\check{P}_\varphi(z)}] \\
&\quad = \overline{\left[\frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}\right]}\,B_\varphi(z)\,B_\varphi(-z) \\
&\qquad \times [(-z)^{-2N_\varphi+1}P_\varphi(z)\,P_\varphi(-z) + z^{-2N_\varphi+1}P_\varphi(-z)\,P_\varphi(z)] \\
&\quad = 0.
\end{aligned}$$

That is, $I_k = 0$ for all $k \in \mathbb{Z}$, so that

$$\{\eta(\cdot - n): n \in \mathbb{Z}\} \subset W_0.$$

Thirdly, to show that this set is a basis of W_0, we will do even more by determining the decomposition sequence that yields $V_1 = V_0 \oplus W_0$. The rational function

$$G_\varphi(z) := z^{-N_\varphi + k_\varphi} \frac{\Pi_\varphi(z)\, \check{P}_\varphi(z)}{\Pi_\varphi(z^2)} = \frac{\beta_\varphi(z)}{z^{2N_\varphi - 2k_\varphi - 1} \Pi_\varphi(z^2)} \tag{4.11}$$

is instrumental for determining this sequence. Also, for this η under consideration, we set

$$H_\eta(z) := \frac{-P_\varphi(-z)\, \lambda_\varphi(z^2)}{z^{2N_\varphi - 2k_\varphi - 1} \Pi_\varphi(z^2)} \cdot \frac{1}{w_\eta(z^2)}. \tag{4.12}$$

The reason for this choice of H_η is to establish the pair of identities

$$\begin{aligned} P_\varphi(z)\, G_\varphi(z) + Q_\eta(z)\, H_\eta(z) &= 1 \\ P_\varphi(z)\, G_\varphi(-z) + Q_\eta(z)\, H_\eta(-z) &= 0, \end{aligned} \tag{4.13}$$

where the idea originates from our previous work in [5, 6]. The proof of (4.13) follows by applying the identity (2.6) in Theorem 2.1 and referring to (4.2), (4.3), and (4.4) in [6]. Now, by setting

$$\begin{aligned} A_1(z^2) &:= G_\varphi(z) + G_\varphi(-z) \\ B_1(z^2) &:= H_\eta(z) + H_\eta(-z), \end{aligned} \tag{4.14}$$

we have, from (4.13), the identity

$$P_\varphi(z)\, A_1(z^2) + Q_\eta(z)\, B_1(z^2) = 1. \tag{4.15}$$

Hence, for $z = e^{-i\omega/2}$ and applying the two-scale formulas (1.4) and (4.1) (see, for instance (4.10)), we have

$$\begin{aligned} \hat{\varphi}\left(\frac{\omega}{2}\right) &= A_1(z^2)\left[P_\varphi(z)\, \hat{\varphi}\left(\frac{\omega}{2}\right)\right] + B_1(z^2)\left[Q_\eta(z)\, \hat{\varphi}\left(\frac{\omega}{2}\right)\right] \\ &= A_1(z^2)\, \hat{\varphi}(\omega) + B_1(z^2)\, \hat{\eta}(\omega). \end{aligned} \tag{4.16}$$

In addition, let us set

$$\begin{aligned} A_2(z^2) &:= z[G_\varphi(z) - G_\varphi(-z)] \\ B_2(z^2) &:= z[H_\eta(z) - H_\eta(-z)], \end{aligned} \tag{4.17}$$

so that from (4.13), we have the identity

$$P_\varphi(z)\, A_2(z^2) + Q_\eta(z)\, B_2(z^2) = z.$$

Hence, as above, it also follows that, for $z = e^{-i\omega/2}$,

$$z\hat{\varphi}\left(\frac{\omega}{2}\right) = A_2(z^2)\, \hat{\varphi}(\omega) + B_2(z^2)\, \hat{\eta}(\omega). \tag{4.18}$$

The pair of equations (4.16) and (4.18) yields the necessary decomposition formula, provided that $A_1(z^2)$, $A_2(z^2)$, $B_1(z^2)$, and $B_2(z^2)$ are all in $L^2(|z| = 1)$. Indeed, by setting

$$\begin{aligned} A_1(z) &= 2 \sum_{n\in\mathbb{Z}} a_{-2n} z^n \\ A_2(z) &= 2 \sum_{n\in\mathbb{Z}} a_{1-2n} z^n \\ B_1(z) &= 2 \sum_{n\in\mathbb{Z}} b_{-2n} z^n \\ B_2(z) &= 2 \sum_{n\in\mathbb{Z}} b_{1-2n} z^n, \end{aligned} \tag{4.19}$$

we see that the pair of (4.16) and (4.18) together is equivalent to

$$\varphi(2x - l) = \sum_{n\in\mathbb{Z}} a_{l-2n} \varphi(x-n) + \sum_{n\in\mathbb{Z}} b_{l-2n} \eta(x-n) \tag{4.20}$$

for all $l \in \mathbb{Z}$.

Now, returning to the consideration of the four functions in (4.19), we first observe that from Theorem 2.1(ii), the rational function G_φ in (4.11) is pole-free on $|z| = 1$, so that A_1 and A_2 are analytic in a neighborhood of $|z| = 1$ (cf. (4.14) and (4.17)). On the other hand, from the assumption that ω_η is in the class $\mathscr{A}$, we may conclude, by applying Theorem 2.1(ii) and Remark 4.4, that

$$\frac{1}{\Pi_\varphi w_\varphi} \in \mathscr{A};$$

and hence, since P_φ and λ_φ are polynomials, we see that H_η, and consequently B_1 and B_2, is also analytic in a neighborhood of $|z| = 1$.

Therefore, the decomposition formula (4.20) is defined by two l^2-sequences $\{a_n\}$ and $\{b_n\}$, which are actually of exponential decay. We will return to the discussion of wavelet decompositions in the next section. At this point, we observe that a trivial consequence of the decomposition formula (4.20) is that

$$\{\eta(\cdot - n) : n \in \mathbb{Z}\}$$

is a basis of W_0. So, to prove that $\eta \in \Psi$, it is sufficient to verify that this basis is unconditional, and in view of Remark 2.3, it is sufficient to establish (2.11) for $\hat{\eta}$.

Now, by Theorem 2.1(i), (iv) and the two-scale formula (4.1), we have, for $z = e^{-i\omega/2}$,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} |\hat{\eta}(\omega+2\pi n)|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{\omega}{2}+2\pi n\right)\right|^2 |Q_\eta(z)|^2 + \sum_{n\in\mathbb{Z}} \left|(\hat{\varphi}\left(\frac{\omega}{2}+\pi+2\pi n\right)\right|^2 |Q_\eta(-z)|^2 \\
&= B_\varphi(z)\,|Q_\eta(z)|^2 + B_\varphi(-z)\,|Q_\eta(-z)|^2 \\
&= [B_\varphi(z)\,|B_\varphi(-z)|^2\,|P_\varphi(-z)|^2 + B_\varphi(-z)\,|B_\varphi(z)|^2\,|P_\varphi(z)|^2] \left|\frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}\right|^2 \\
&= B_\varphi(z)\,B_\varphi(-z)[|P_\varphi(-z)|^2\,B_\varphi(-z) + |P_\varphi(z)|^2\,B_\varphi(z)] \left|\frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}\right|^2 \\
&= B_\varphi(z)\,B_\varphi(-z)\,B_\varphi(z^2) \left|\frac{w_\eta(z^2)}{\lambda_\varphi(z^2)}\right|^2 \\
&= \left|\frac{\Pi_\varphi(z)\,\Pi_\varphi(-z)\,\Pi_\varphi(z^2)(w_\eta(z^2))^2}{(\lambda_\varphi(z^2))^2}\right|.
\end{aligned}$$

Here, since both the numerator and the denominator of the last quantity are analytic and zero-free on $|z|=1$ (cf. Remark 4.3), this quantity is continuous and hence, bounded below and above on all $|z|=1$.

(ii) Conversely, suppose that $\eta \in \Psi$. We must establish (4.4) for some $w_\eta \in \mathscr{A}$. Let ψ be defined by setting $\omega_\psi(z)=1$, or equivalently, $Q_\psi(z)=\mu_\varphi(-z)$ in (4.4). Hence, by what we just proved, we have $\psi \in \Psi$. This yields, in particular,

$$\hat{\eta}(\omega) = C(z^2)\,\psi(\omega), \qquad z = e^{-i\omega/2},$$

where $C(z^2)$ is in $L^2(|z|=1)$. By the same proof as that of the converse of Theorem 3.1, we also note that $C(z) \neq 0$ on $|z|=1$.

On the other hand, since $\eta \in \Psi$, we have

$$\hat{\eta}(\omega) = Q_\eta(z)\varphi\left(\frac{\omega}{2}\right), \qquad z = e^{-i\omega/2}.$$

Hence, combining the above formulas, we have

$$Q_\eta(z)\varphi\left(\frac{\omega}{2}\right) = C(z^2)\,\psi(\omega) = C(z^2)\,Q_\psi(z)\varphi\left(\frac{\omega}{2}\right),$$

so that

$$Q_\eta(z) = \mu_\varphi(-z)\, C(z^2).$$

That is, $w_\eta(z) = C(z^2)$. We have already seen that $C(z) \neq 0$ on $|z| = 1$. Since $C(z^2)$ is also in $L^2(|z| = 1)$, the quotient $Q_\eta(z)/\mu_\varphi(-z)$, being analytic on $|z| = 1$ except at the location of the possible zeros of the polynomial $\mu_\varphi(-z)$, cannot have any poles on $|z| = 1$. This shows that $w_\eta(z) = C(z^2)$ is in $\mathscr{A}$, and completes the proof of the theorem. ∎

5. Algorithms and Linear-Phase Filtering

Let φ be the (generalized) B-spline that generates a given multiresolution analysis and choose any wavelet $\eta \in \Psi$. Although in most applications, we prefer the B-wavelet $\psi \in \Psi$ whose minimum support facilitates both computational and implementational efficiency, we will see that a "symmetric" wavelet η is important in filtering applications, since "symmetry" is required for the filtering process to have *linear phase*.

We first return to the formulation of the decomposition algorithm (4.20), with decomposition sequences $\{a_n\}$ and $\{b_n\}$. Let G_φ and H_η be as defined in (4.11) and (4.12), respectively. Then from the definitions of A_1, A_2, B_1, B_2 in (4.14) and (4.17), it is clear that (4.19) is equivalent to

$$\begin{aligned} G_\varphi(z) &= \sum_{n \in \mathbb{Z}} a_n z^{-n} \\ H_\eta(z) &= \sum_{n \in \mathbb{Z}} b_n z^{-n}. \end{aligned} \tag{5.1}$$

Hence, the expressions of G_φ and H_η in terms of Π_φ, P_φ, λ_φ, and w_η are important in yielding the decomposition sequence $\{a_n\}$ and $\{b_n\}$.

Example 5.1. Let φ be the polynomial B-spline N_m of order m and ψ_m be the corresponding B-wavelet given in (4.9). Then

$$\begin{aligned} G_{N_m}(z) &= \left(\frac{1+z}{2}\right)^m \frac{\Pi_{2m-1}(z)}{z\Pi_{2m-1}(z^2)} \\ H_{\psi_m}(z) &= -(2m-1)!\left(\frac{1-z}{2}\right)^m \frac{1}{z\Pi_{2m-1}(z^2)}, \end{aligned} \tag{5.2}$$

which were derived in [6]. So, if $r_1, ..., r_{2m-2}$ denote the roots of the $(2m-2)$nd degree Euler–Frobenius polynomial $\Pi_{2m-1}(z)$, where

$$r_{2m-2} < r_{2m-3} < \cdots < r_m < -1 < r_{m-1} < \cdots < r_1 < 0$$

and $r_1 r_{2m-2} = r_2 r_{m-3} \cdots = r_{m-1} r_m = 1$, then the rate of decay of the decomposition sequences $\{a_n\}$ and $\{b_n\}$ is $O(|r_m|^{-|n|/2})$ as $|n| \to \infty$.

Remark 5.1. If φ happens to be o.n. as in [9] in the sense of (11°), then by Corollary 4.3, the B-wavelet ψ is also o.n., and we have

$$\begin{aligned} P_\varphi(z) &= \frac{1}{2} \sum_{n=0}^{N_\varphi} p_n^\varphi z^n \\ Q_\psi(z) &= \frac{1}{2} \sum_{n=\varepsilon_\varphi}^{N_\varphi+\varepsilon_\varphi} (-1)^n \overline{p_{N_\varphi-n+\varepsilon_\varphi}^\varphi} z^n, \end{aligned} \tag{5.3}$$

where $\varepsilon_\varphi = 0$ for odd N_φ and $\varepsilon_\varphi = 1$ for even N_φ as defined in the statement of Corollary 4.3. Hence, the pair of two-scale (reconstruction) sequences is given by

$$\begin{aligned} p_n &= p_n^\varphi, \qquad n = 0, ..., N_\varphi \\ q_n = q_n^\psi &= (-1)^n \overline{p_{N_\varphi-n+\varepsilon_\varphi}^\varphi}, \qquad n = \varepsilon_\varphi, ..., N_\varphi + \varepsilon_\varphi, \end{aligned} \tag{5.4}$$

where, as usual, the undefined terms p_n and q_n are set to be zero.

Note that the orthogonality condition also implies that $\Pi_\varphi(z) = 1$, $k_\varphi = 0$, $\lambda_\varphi(z^2) = z^{N_\varphi - 1 - \varepsilon_\varphi}$, and

$$Q_\psi(z) = (-z)^{\varepsilon_\varphi} \check{P}_\varphi(-z)$$

which agrees with $-z^{N_\varphi+\varepsilon_\varphi} \overline{P_\varphi(-z)}$ for $|z| = 1$. This yields the formulas

$$\begin{aligned} G_\varphi(z) &= z^{-N_\varphi} \check{P}_\varphi(z) \\ H_\psi(z) &= -z^{-N_\varphi-\varepsilon_\varphi} P_\varphi(-z) \end{aligned} \tag{5.5}$$

so that the decomposition sequences are given by

$$\begin{aligned} a_n &= \tfrac{1}{2} \overline{p_n^\varphi}, \qquad && n = 0, ..., N_\varphi \\ b_n &= \tfrac{1}{2}(-1)^n p_{N_\varphi-n+\varepsilon_\varphi}^\varphi, \qquad && n = \varepsilon_\varphi, ..., N_\varphi + \varepsilon_\varphi, \end{aligned} \tag{5.6}$$

which are both finite sequences. Observe that since $\check{P}_\varphi(z) = z^{N_\varphi} \overline{P_\varphi(z)}$ for $|z| = 1$, the pair of identities in (4.13) now becomes a single identity

$$|P_\varphi(z)|^2 + |P_\varphi(-z)|^2 = 1, \qquad |z| = 1,$$

often used in [9, 14].

In general, for any multiresolution analysis with (generalized) B-spline $\varphi \in \Phi$ and an arbitrary wavelet $\eta \in \Psi$, let us recall the notations in (1.3) and (1.13), namely

$$\{\varphi_{k,j}\}_{j \in \mathbb{Z}}, \qquad \{\eta_{k,j}\}_{j \in \mathbb{Z}} \tag{5.7}$$

which are unconditional bases of V_k and W_k, respectively. Then from the discussion in Section 1, it is obvious that every function $f_N \in V_N$, where $N \in \mathbb{Z}$, has a (unique) orthogonal decomposition

$$f_N = g_{N-1} \oplus \cdots \oplus g_{N-M} \oplus f_{N-M} \tag{5.8}$$

for any $M > 0$, where $g_k \in W_k$ and $f_{N-M} \in V_{N-M}$. Hence, by expressing each of these orthogonal components in terms of the unconditional bases (5.7), namely

$$\begin{aligned} f_k(x) &= \sum_{j \in \mathbb{Z}} c_j^k \varphi_{k,j}(x) \\ g_k(x) &= \sum_{j \in \mathbb{Z}} d_j^k \eta_{k,j}(x), \end{aligned} \tag{5.9}$$

it is not difficult to see that the coefficient sequences

$$\begin{aligned} \mathbf{c}^k &= \{c_j^k\}, \qquad j \in \mathbb{Z} \\ \mathbf{d}^k &= \{d_j^k\}, \qquad j \in \mathbb{Z} \end{aligned} \tag{5.10}$$

satisfy both the *decomposition relation*

$$\begin{aligned} c_j^{k-1} &= \sum_{n \in \mathbb{Z}} a_{n-2j} c_n^k, \qquad j \in \mathbb{Z} \\ d_j^{k-1} &= \sum_{n \in \mathbb{Z}} b_{n-2j} c_n^k, \qquad j \in \mathbb{Z} \end{aligned} \tag{5.11}$$

and the *reconstruction relation*

$$c_j^k = \sum_{n \in \mathbb{Z}} p_{j-2n} c_n^{k-1} + \sum_{n \in \mathbb{Z}} q_{j-2n} d_n^{k-1}, \qquad j \in \mathbb{Z}, \tag{5.12}$$

where we have used the notation

$$\begin{aligned} p_n &:= \begin{cases} p_n^{\varphi}, & n = 0, ..., N_{\varphi} \\ 0, & \text{otherwise} \end{cases} \\ q_n &:= q_n^{\eta}. \end{aligned} \tag{5.13}$$

Indeed, noting that for any nontrivial function $\xi \in L^1 \cap L^2$, the assumption

$$\sum_{j \in \mathbb{Z}} \alpha_j \xi(\cdot - j) = 0 \qquad \text{a.e., } \{\alpha_j\} \in l^2,$$

implies that $\alpha_j = 0$ for all j, and applying the orthogonality property $\varphi_{k,j} \perp \eta_{k,l}$ for all $j, l \in \mathbb{Z}$, the relation

$$f_k(x) = f_{k-1}(x) + g_{k-1}(x)$$

can be shown to be equivalent to (5.11) by the decomposition formula (4.20), and equivalent to (5.12) by the pair of two-scale formulas (1.4) and (4.1). For this reason, the two-scale sequences $\{p_n\}$ and $\{q_n\}$ in (5.13) are also called reconstruction sequences. That is, we have the following so-called *pyramid algorithms* (cf. [15, 9]).

Algorithm 1 (Decomposition Algorithm). The orthogonal wavelet decomposition (5.8) can be realized by applying the decomposition relation (5.11) recursively, as follows:

$$\begin{tikzcd}[column sep=small] & \mathbf{d}^{N-1} & & \mathbf{d}^{N-2} & & & & \mathbf{d}^{N-M} \\ \mathbf{c}^{N} \arrow[r] \arrow[ur] & \mathbf{c}^{N-1} \arrow[rr] \arrow[urr] & & \mathbf{c}^{N-2} \arrow[r] \arrow[urr] & \cdots \arrow[rrr] \arrow[urrr] & & & \mathbf{c}^{N-M} \end{tikzcd}$$

Algorithm 2 (Reconstruction Algorithm). The reconstruction of f_N from its orthogonal components in (5.8) can be realized by applying the reconstruction relation (5.12) recursively, as follows:

$$\begin{tikzcd}[column sep=small] \mathbf{d}^{N-M} \arrow[dr] & \mathbf{d}^{N-M+1} \arrow[dr] & & & \mathbf{d}^{N-1} \arrow[dr] & \\ \mathbf{c}^{N-M} \arrow[r] & \mathbf{c}^{N-M+1} \arrow[r] & \cdots \arrow[rr] & & \mathbf{c}^{N-1} \arrow[r] & \mathbf{c}^{N} \end{tikzcd}$$

Remark 5.2. We have seen in Remark 5.1 that if both $\varphi \in \Phi$ and $\eta \in \Psi$ are o.n. in the sense of (11°), then all the decomposition sequences $\{a_n\}$, $\{b_n\}$ and reconstruction sequences $\{p_n\}$, $\{q_n\}$ are finite sequences (cf. (5.6) and (5.4)). In the general setting, by choosing $\eta \in \Psi_p$, although the reconstruction sequences $\{p_n\}$ and $\{q_n\}$ are both finite sequences, we note that $\{a_n\}$ cannot be finite if φ is not o.n. and $\{b_n\}$ cannot be finite unless both φ is o.n. and η has minimum support. The reason is that from (4.11) and (4.12), for G_φ and H_η to be in π, it is necessary and sufficient that both Π_φ and w_η are constants. Hence, to implement non-o.n. $\varphi \in \Phi$ and $\eta \in \Psi$, by applying the pyramid algorithms, such as the polynomial B-splines N_m and B-wavelets ψ_m, $m \geqslant 2$, the decomposition sequences have to be truncated. We will establish a *duality principle* in the next section to essentially interchange the pair of decomposition sequences and the pair of reconstruction sequences, so that even for the polynomial spline setting, the decomposition sequences can still be finite (cf. [6]). The importance of using polynomial splines and wavelets for linear filtering, over any o.n. φ and ψ, such as those constructed in the celebrated paper [9], is that both polynomial B-splines and B-wavelents have (at least generalized) *linear phases*, while due to their nonsymmetric behavior (cf. [9]), o.n. φ and ψ do not have this important property, and thus may cause phase distortion of the signals (cf. [18, pp. 250–269]). When we study linear phases, we will always assume, for notational convenience, that all the functions under consideration are real-valued functions.

DEFINITION 5.1. A real-valued function $f \in L^1$ is said to have *generalized linear phase* if its Fourier transform has the representation

$$\hat{f}(\omega) = A(\omega)\, e^{i(a\omega + b)}, \qquad \omega \in \mathbb{R}, \tag{5.14a}$$

where a and b are real constants and $A(\omega)$ is a real-valued function.

DEFINITION 5.2. A real-valued function $f \in L^1$ is said to have *linear phase* if its Fourier transform satisfies

$$\hat{f}(\omega) = \pm |\hat{f}(\omega)|\, e^{ia\omega}, \qquad \omega \in \mathbb{R}, \tag{5.14b}$$

for some real constant a, where the $+$ or $-$ sign is independent of ω.

EXAMPLE 5.2. Let N_m and ψ_m denote the mth order polynomial B-spline and B-wavelet (cf. (4.9)). Then from Example 4.1 we see that

$$\hat{N}_m(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^m e^{-i(m/2)\omega} \tag{5.15a}$$

and

$$\hat{\psi}_m(\omega) = \frac{1}{(2m-1)!}\left(\frac{4}{\omega}\right)^m \left|\left(\sin\frac{\omega}{4}\right)^{2m} \Pi_{2m-1}(-e^{-i(\omega/2)})\right| \times e^{i[(-m+1/2)\omega + ((3m-2)/2)\pi]}. \tag{5.15b}$$

Hence, both N_m and ψ_m have generalized linear phases for all m, and have linear phases for all even m. For this reason, even-order polynomial splines and wavelets are more useful in signal processing than the odd-order ones, although generalized linear phases can also be handled by standard methods (cf. [18, pp. 250–269]).

It is well known in the signal processing literature (cf. [18, pp. 250–269]) that all symmetric or antisymmetric functions have generalized linear phases. In fact, the converse also holds as in the following.

LEMMA 5.1. *A real-valued function $f(x)$ has generalized linear phase if and only if it is either symmetric or antisymmetric about some $x_0 \in \mathbb{R}$.*

Remark 5.3. In view of this lemma and the nonsymmetric property established by Daubechies [9], all o.n. (compactly supported) φ and ψ (except N_1 and the Haar function ψ_1), do *not* have generalized linear phases.

Proof. (i) If $f(x)$ has generalized linear phase, then there is a real-valued function $A(\omega)$ such that

$$\hat{f}(\omega)\, e^{i(x_0\omega - b)} = A(\omega), \qquad \omega \in \mathbb{R},$$

for some real constants x_0 and b. Taking the complex conjugates of both sides, we have

$$\hat{f}(\omega)\, e^{i(x_0\omega - b)} = \overline{\hat{f}(\omega)}\, e^{-i(x_0\omega - b)}.$$

Hence, since $f(x)$ is real-valued, the inverse transform yields

$$\begin{aligned} f(x_0 + x) &= e^{i2b}\, \overline{f(x_0 - x)} \\ &= e^{i2b} f(x_0 - x) = \pm f(x_0 - x). \end{aligned}$$

Here, since f is real-valued, we must have $e^{i2b} = \pm 1$.

(ii) The proof of the converse is similar. ∎

To study the phase properties of $\phi \in \Phi$ and $\eta \in \Psi$, we also need the analogous notions of generalized linear phase and linear phase of a real-valued l^1-sequence, as follows:

DEFINITION 5.3. Let $\{f_n\}$ be a real-valued l^1-sequence with "symbol"

$$F(z) := \frac{1}{2} \sum_{n \in \mathbb{Z}} f_n z^n. \tag{5.16}$$

Then $\{f_n\}$ is said to have generalized linear phase if $F(\omega) = A(\omega)\, e^{i(a\omega + b)}$ for some real-valued function $A(\omega)$ and real constants a and b. It is said to have linear phase if $F(\omega) = \pm |F(\omega)|\, e^{ia\omega}$ for some real constant a, where the $+$ or $-$ sign is independent of ω.

Analogous to Lemma 5.1, we also have the following.

LEMMA 5.2. *A real-valued l^1-sequence has generalized linear phase if and only if it is either symmetric or antisymmetric with respect to some $n_0 \in \frac{1}{2}\mathbb{Z}$; that is, $f_n = \pm f_{2n_0 - n}$, for all $n \in \mathbb{Z}$.*

Remark 5.4. The "symmetric" condition in Lemma 5.2 provides a standard tool in digital filter design (cf. [18, pp. 465–488]) to achieve generalized linear-phase filtering. That the point n_0 of symmetry or antisymmetry must be in $\frac{1}{2}\mathbb{Z}$ follows from the 2π-periodicity of $F(e^{i\omega})$. Otherwise, the proof of Lemma 5.2 is the same as that of Lemma 5.1.

Remark 5.5. If $\operatorname{supp} f = [c, d]$, then the center $x_0 = (c + d)/2$ is the point of symmetry or antisymmetry for a generalized linear-phase f, and

$\hat{f}(\omega) = \gamma A(\omega) e^{-ix_0\omega}$ for some constant γ with $\gamma^2 = \pm 1$. Similarly, if $\{f_n\}$ is finite with $\operatorname{supp}\{f_n\} = [k_1, k_2] \cap \mathbb{Z}$ $(k_1, k_2 \in \mathbb{Z})$, and has generalized linear phase, then the center of symmetry or antisymmetry of $\{f_n\}$ is $n_0 = (k_1 + k_2)/2 \in \frac{1}{2}\mathbb{Z}$, and $F(e^{i\omega}) = \gamma A(\omega) e^{in_0\omega}$, with $\gamma^2 = \pm 1$.

Characterization of (strict) linear phase is similar. Let us first study these properties for sequences.

LEMMA 5.3. *A real-valued l^1-sequence $\{f_n\}$ has linear phase if and only if there exists some $n_0 \in \mathbb{Z}$ such that the function $F(e^{i\omega}) e^{-in_0\omega}$ is real-valued, even, and has no sign changes.*

Proof. By definition, $\{f_n\}$ has linear phase if and only if $F(e^{i\omega}) e^{-ia\omega}$ is real-valued and does not change sign, where a is some real constant. This constant a must be an integer n_0 because of the 2π-periodicity of $F(e^{i\omega})$. That $F(e^{i\omega}) e^{-in_0\omega}$ is even follows from the assumption that $\{f_n\}$ is a real sequence. ▮

If the sequence $\{f_n\}$ is finite, we can say a little more.

LEMMA 5.4. *A real-valued sequence $\{f_n\}$ with support $[0, N]$ has linear phase if and only if $N = 2N'$ with $N' \in \mathbb{Z}$, F is a polynomial whose roots on $|z| = 1$ have even orders, and $\check{F} = F$.*

Proof. The proof involves factorizing F into $F = F_1 F_2$ where F_1 has all its roots on $|z| = 1$ and F_2 does not vanish on $|z| = 1$, observing that $\check{F}_1 = F_1$, $\check{F}_2 = F_2$, $\deg F_1$ and $\deg F_2$ are both even, and applying Lemma 5.3. ▮

This argument also applies to infinite $\{f_n\}$, provided that F has an analogous factorization. Since it will be useful for studying the phase property of all dual bases, we formulate this result in the following remark.

Remark 5.6. Let $\{f_n\} \in l^1$ be real-valued such that its symbol F has the representation $F = F_1 R$ where $F_1 \in \pi$ with all of its zeros lying on $|z| = 1$ and $R(e^{i\omega}) \neq 0$ for all ω. Then $\{f_n\}$ has linear phase if and only if $F(e^{i\omega}) e^{-in_0\omega}$ is real-valued for some $n_0 \in \mathbb{Z}$ and all the zeros of F_1 have even multiplicities.

We are now ready to study the phase properties of (generalized) B-splines and B-wavelets. We note, however, that since we will also discuss the phase properties for the dual bases in the next section, it is necessary to include those multiresolution analysis generators without compact supports. As before, let us restrict our attention to a fixed multiresolution analysis generated by an $\phi \in \Phi$. Let $\tilde{\Phi}$ denote the collection of all ϕ where each ϕ generates the same multiresolution analysis such that $\hat{\phi}(0) = 1$, $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is an unconditional basis of V_0, and ϕ satisfies a two-scale formula whose defining sequence $\{p_n^\phi\}$ is in l^1 but may not be finite. We will again denote its "symbol" by P_ϕ (recalling that a factor of $\frac{1}{2}$ is used). Note that $\tilde{\Phi} \supset \Phi$. In studying the phase properties, we will assume, for convenience, that all the two-scale sequences $\{p_n^\phi\}$ and $\{q_n^\eta\}$ are real-valued.

Remark 5.7. Under the assumption that $\{p_n^\phi\} \in l^1$ is real-valued and the condition that $\hat{\phi}(0) = 1$, where $\phi \in \tilde{\Phi}$, it is clear that if ϕ has generalized linear phase, then we may write $\hat{\phi}(\omega) = A_\phi(\omega)\, e^{ia\omega}$ with real-valued function A_ϕ and real constant a. That is, the shift by b in Definition 5.1 necessarily disappears.

We have the following result.

THEOREM 5.1. *Let $\phi \in \tilde{\Phi}$ be defined by a real-valued l^1 two-scale sequence with symbol P_ϕ. Then*

(i) *ϕ has generalized linear phase if and only if $P_\phi(e^{i\omega}) = A(\omega)\, e^{in_0\omega}$ where $A(\omega)$ is real-valued and $n_0 \in \frac{1}{2}\mathbb{Z}$; and*

(ii) *ϕ has linear phase if and only if*

$$P_\phi(e^{i\omega}) = |P_\phi(e^{i\omega})|\, e^{in_0\omega}, \qquad n_0 \in \tfrac{1}{2}\mathbb{Z}.$$

Proof. By Lemma 5.2 and Remark 5.7, we note that for $\{p_n^\phi\}$ to have generalized linear phase, the two-scale formula yields, for $z = e^{-i\omega/2}$,

$$P_\phi(z) = \frac{\hat{\phi}(\omega)}{\hat{\phi}(\omega/2)} = \frac{e^{i2a\omega}\hat{\phi}(-\omega)}{e^{ia\omega}\hat{\phi}(-\omega/2)} = e^{ia\omega}\,\overline{P_\phi(z)},$$

so that $P_\phi(e^{i\omega})\, e^{ia\omega} =: A(\omega)$ is real-valued. Here, by the 2π-periodicity of $P_\phi(e^{i\omega})$, we have $-a =: n_0 \in \frac{1}{2}\mathbb{Z}$. For the linear-phase setting, we even have $n_0 \in \mathbb{Z}$ by Lemma 5.3 and, with $z = e^{-i\omega/2}$, it follows that

$$P_\phi(z)\, z^{n_0} = \overline{P_\phi(z)\, z^{n_0}} = \frac{e^{-in_0\omega}\hat{\phi}(\omega)}{e^{-i(n_0/2)\omega}\hat{\phi}(\omega/2)} \geqslant 0,$$

since for linear-phase ϕ, both the numerator and the denominator are non-negative.

To verify the converse, we rely on the infinite product formulation of $\hat{\phi}$ with $\hat{\phi}(0) = 1$, namely, if $P_\phi(e^{i\omega})\, e^{-in_0\omega} = A(\omega)$ is real, then

$$\begin{aligned}
\hat{\phi}(\omega) &= \prod_{j=1}^{\infty} P_\phi(e^{-i\omega/2^j}) \\
&= \prod_{j=1}^{\infty} [P_\phi(e^{i\omega/2^j})\, e^{-i2n_0\omega/2^j}] \\
&= e^{-i2n_0\omega}\hat{\phi}(-\omega)
\end{aligned}$$

which is equivalent to $\phi(n_0 + x) = \phi(n_0 - x)$. ∎

If $\phi \in \Phi$ so that $\{p_n^{\phi}\}$ is a (real-valued) finite sequence, then with the aid of Lemma 5.4, the above argument also gives the following result.

THEOREM 5.2. *Let $\phi \in \Phi$ be defined by a real-valued (finite) two-scale sequence. Then*

(i) *ϕ has generalized linear phase if and only if $\check{P}_\phi = P_\phi$; and*

(ii) *ϕ has linear phase if and only if $\check{P}_\phi = P_\phi$ and all the zeros of P_ϕ on the unit circle, if any, have even multiplicities.*

Recall that for $\varphi \in \Phi$, the corresponding (minimally supported) B-wavelet $\psi \in \Psi_p$ satisfies the two-scale formula (4.1) with polynomial "symbol"

$$Q_\psi(z) = \mu_\varphi(-z) = \frac{1}{2} \sum_{n \in \mathbb{Z}} q_n^\psi z^n,$$

where μ_φ is defined in (4.3) (here, also recall the factor $\lambda_\varphi(z^2)$ of β_φ in (4.2) and (4.3)). We have the following result on the phase property of B-wavelets.

THEOREM 5.3. *Let $\varphi \in \Phi$ be defined by a real-valued (finite) two-scale sequence $\{p_n^\varphi\}$. Then*

(i) *if $\{p_n^\varphi\}$ has generalized linear phase, the wavelet ψ also has generalized linear phase; and*

(ii) *if $\{p_n^\varphi\}$ has linear phase, the wavelet ψ also has linear phase.*

Proof. In proving this theorem, we must also show that the two-scaled sequence $\{q_n^\psi\}$ for ψ is also real-valued. The key idea in this proof is that since λ_φ is zero-free on $|z| = 1$ by Remark 4.3, and since $(\Pi_\varphi P_\varphi)^\vee = \Pi_\varphi P_\varphi$ provided $\check{P}_\varphi = P_\varphi$, we have $\lambda_\varphi(z_0^2) = 0$ if and only if $\lambda_\varphi(z_0^{-2}) = 0$ for $z_0 \neq 0$, and $\mu_\varphi(z_0) = 0$ if and only if $\mu_\varphi(z_0^{-1}) = 0$ for $z_0 \neq 0$. This yields, using the normalization $z^2 \nmid \mu_\varphi(z)$,

$$\mu_\varphi(z) = z^{\varepsilon_\varphi} \check{\mu}_\varphi(z)$$

(recall the definition of ε_φ in Corollary 4.3). So, if $\{p_n^\varphi\}$ has generalized linear phase (or linear phase), $\{q_n^\psi\}$ is real and also has generalized linear phase (or linear phase). Hence, we can apply Theorem 5.2 to complete the proof of this theorem. ∎

6. Dual Bases and a Duality Principle

For $\varphi \in \Phi$ and any $\eta \in \Psi_p$, both of the two-scale (or reconstruction) sequences in (5.13) are finite. However, unless φ and $\eta = \psi$ are both o.n. in the sense of (11°), the two decomposition sequences must be infinite, although they are of exponential decay. For strictly (generalized) linear-phase considerations, however, o.n. φ and ψ (different from the first order B-spline N_1 and Haar function ψ_1) cannot be used, and it seems that the only way out is either to truncate the decomposition sequences or to go to non-orthogonal wavelet decomposition. We will only briefly discuss the second option considered recently in [8, 11] by introducing the so-called "biorthogonal wavelets" later in this section. Our approach in this section is to "interchange" the finite reconstruction sequences with the decomposition sequences. This idea, initiated in [6] for the polynomial spline and wavelet setting, is valid in general due to the symmetry of the "two-scale polynomials," P_φ and Q_η, and the "reconstruction polynomials," G_φ and H_η, in the identities (4.13). The technique is to consider the *duals* of φ and η defined as follows.

DEFINITION 6.1. $\tilde{\varphi} \in \tilde{\Phi}$ is said to be *dual* to φ (or equivalently $\{\tilde{\varphi}(\cdot - n) : n \in \mathbb{Z}\}$ is dual to $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$) if

$$\langle \tilde{\varphi}(\cdot - m), \varphi(\cdot - n) \rangle = \int_{-\infty}^{\infty} \tilde{\varphi}(x - m)\, \overline{\varphi(x - m)}\, dx = \delta_{m,n}.$$

The same formulation applies to defining the dual $\tilde{\eta} \in \Psi_p$ of η.

Remark 6.1. Since $\{\varphi(\cdot - n) : n \in Z\}$ is an unconditional basis of V_0, the dual $\tilde{\varphi}$ in V_0 of φ is unique. Similarly, any $\eta \in \Psi_p$ has a unique dual $\tilde{\eta} \in \Psi_p$.

We have the following result.

THEOREM 6.1. *The Fourier transforms $\hat{\tilde{\varphi}}$ and $\hat{\tilde{\eta}}$ of the duals $\tilde{\varphi} \in \Phi$ and $\tilde{\eta} \in \Psi_p$ of φ and η, respectively, satisfy the two-scale formulas*

$$\hat{\tilde{\varphi}}(\omega) = \overline{G_\varphi(z)}\, \hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right), \qquad z = e^{-i\omega/2}, \tag{6.1}$$

$$\hat{\tilde{\eta}}(\omega) = \overline{H_\eta(z)}\, \hat{\tilde{\eta}}\left(\frac{\omega}{2}\right), \qquad z = e^{-i\omega/2}. \tag{6.2}$$

Remark 6.2. There is a more direct way to express $\tilde{\varphi}$ in terms of φ, and $\tilde{\eta}$ in terms of η. In fact, we have the identities

$$\hat{\tilde{\varphi}}(\omega) = \frac{\hat{\varphi}(\omega)}{\sum_{n \in \mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2}, \tag{6.3}$$

$$\hat{\tilde{\eta}}(\omega) = \frac{\hat{\eta}(\omega)}{\sum_{n \in \mathbb{Z}} |\hat{\eta}(\omega + 2\pi n)|^2}, \tag{6.4}$$

where the denominators are strictly positive since we have unconditional bases (cf. Remark 2.3). To verify (6.3) and (6.4), we simply note that for any $n \in \mathbb{Z}$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(x)\,\overline{\tilde{\varphi}(x-n)}\,dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)\,\overline{\hat{\tilde{\varphi}}(\omega)}\,e^{in\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)\,\frac{\overline{\hat{\varphi}(\omega)}}{\sum_{n\in\mathbb{Z}}|\hat{\varphi}(\omega+2\pi n)|^2}\,e^{in\omega}\,d\omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)} \frac{|\hat{\varphi}(\omega)|^2}{\sum_{n\in\mathbb{Z}}|\hat{\varphi}(\omega+2\pi n)|^2}\,e^{in\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{in\omega}\,d\omega = \delta_{n,0}.
\end{aligned}$$

We now turn to the proof of Theorem 6.1.

Proof. (i) To prove (6.1), we apply (6.3) in the above remark, and (2.3), (7°), and (4.11) to obtain

$$\begin{aligned}
\hat{\tilde{\varphi}}(\omega) &= \frac{\hat{\varphi}(\omega)}{B_\varphi(z^2)} = \frac{P_\varphi(z)}{B_\varphi(z^2)}\,\hat{\varphi}\left(\frac{\omega}{2}\right) \\
&= \frac{P_\varphi(z)}{B_\varphi(z^2)}\,B_\varphi(z)\,\hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right) \\
&= \frac{z^{N_\varphi}\overline{\check{P}_\varphi(z)}}{z^{2k_\varphi}\overline{\Pi_\varphi(z^2)}}\,z^{k_\varphi}\overline{\Pi_\varphi(z)}\,\hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right) \\
&= \overline{G_\varphi(z)}\,\hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right),
\end{aligned}$$

where $z = e^{-i\omega/2}$.

(ii) To verify (6.2), we take advantage of the uniqueness of $\tilde{\eta}$ and directly compute, for all $n \in \mathbb{Z}$, the inner product

$$\begin{aligned}
I_n &:= \int_{-\infty}^{\infty} \eta(x)\,\overline{\tilde{\eta}(x-n)}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(\omega)\,\overline{\hat{\tilde{\eta}}(\omega)}\,e^{in\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[Q_\eta(z)\,\hat{\varphi}\left(\frac{\omega}{2}\right)\right]\overline{\left[H(z)\,\hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right)\right]}\,e^{in\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (1 - P_\varphi(z)\,G(z))\,\hat{\varphi}\left(\frac{\omega}{2}\right)\overline{\hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right)}\,e^{in\omega}\,d\omega,
\end{aligned}$$

where $z=e^{-i\omega/2}$ and the first identity in (4.13) has been used. Hence, applying (6.1) and duality of $\tilde{\varphi}$ and φ, we have

$$I_n=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\hat{\varphi}(\omega)\,\overline{\hat{\tilde{\varphi}}(\omega)}\,e^{i2n\omega}\,d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)\,\overline{\hat{\tilde{\varphi}}(\omega)}\,e^{in\omega}\,d\omega$$

$$=2\delta_{n,0}-\delta_{n,0}=\delta_{n,0}.$$

This completes the proof of the theorem. ▮

Remark 6.3. Recalling the Laurent expansions of G and H in (5.1), we observe that the two-scale formulas (6.1) and (6.2) are equivalent to

$$\tilde{\varphi}(x)=\sum_{n\in\mathbb{Z}} 2\bar{a}_n\tilde{\varphi}(2x-n) \tag{6.5}$$

and

$$\tilde{\eta}(x)=\sum_{n\in\mathbb{Z}} 2\bar{b}_n\tilde{\varphi}_n(2x-n). \tag{6.6}$$

Hence, with the exception of complex conjugation and a factor of 2, the decomposition sequences $\{a_n\}$ and $\{b_n\}$ are used as the two-scale sequences for the duals $\tilde{\varphi}$ and $\tilde{\eta}$. So, in view of the identities in (4.13), the decomposition sequences for the duals must be $\{\frac{1}{2}\overline{p_n^{\varphi}}\}$ and $\{\frac{1}{2}\overline{q_n^{\eta}}\}$; that is, we have

$$\tilde{\varphi}(2x-l)=\sum_{n\in\mathbb{Z}}\frac{1}{2}\overline{p_{l-2n}^{\varphi}}\tilde{\varphi}(x-n)+\sum_{n\in\mathbb{Z}}\frac{1}{2}\overline{q_{l-2n}^{\eta}}\tilde{\eta}(x-n), \tag{6.7}$$

$l\in\mathbb{Z}$, where both summations are finite.

Hence, we have derived the following.

COROLLARY 6.1 (Duality Principle). *Let $\varphi\in\Phi$ and $\eta\in\Psi_p$, and let $\tilde{\varphi}\in\tilde{\Phi}$ and $\tilde{\eta}\in\Psi_p$ be their corresponding duals. If the two-scale (or reconstruction) and decomposition sequences of (φ,η) are given by $(\{p_n^{\varphi}\},\{q_n^{\eta}\})$, and $(\{a_n\},\{b_n\})$, respectively, then the two-scale (or reconstruction) and decomposition sequences of $(\tilde{\varphi},\tilde{\eta})$ are given by $(\{2\bar{a}_n\},\{2\bar{b}_n\})$ and $(\{\frac{1}{2}\overline{p_n^{\varphi}}\},\{\frac{1}{2}\overline{q_n^{\eta}}\})$, respectively.*

Remark 6.4. If φ is not o.n. in the sense of (11°), then $\tilde{\varphi}$ necessarily has infinite support. Similarly, $\tilde{\eta}$ has infinite support if η is not o.n. Nonetheless, both $\tilde{\varphi}$ and $\tilde{\eta}$ are of exponential decay, and certainly $\tilde{\eta}$ provides a very good wavelet window function for the integral wavelet transform (cf. [10, 13]). The duality of $\tilde{\eta}$ and η now yields a recovery formula of any $f\in L^2$ from its interal wavelet transforms $\langle f,\tilde{\eta}_{k,l}\rangle$, namely

$$f(x)=\sum_{k,l\in\mathbb{Z}} 2^k\langle f,\tilde{\eta}_{k,l}\rangle\,\eta_{k,l}(x), \tag{6.8}$$

where since each $\eta_{k,j}$ has compact support, the summation over $j \in \mathbb{Z}$ in (6.8) is only a finite sum for any fixed $x \in R$. We call (6.8) the *complete wavelet decomposition* of f. Of course, if an approximation of f from V_n is already made, say by $f_N \in V_N$, then we have the following *wavelet decomposition* of f_N as formulated in (5.8), namely

$$f_N(x) = \sum_{k=N-M}^{N-1} \sum_{j \in \mathbb{Z}} 2^k \langle f_N, \tilde{\eta}_{k,j} \rangle \eta_{k,j}(x) + \sum_{j \in \mathbb{Z}} 2^{N-M} \langle f_N, \tilde{\varphi}_{N-M,j} \rangle \varphi_{N-M,j}(x), \tag{6.9}$$

where for each $x \in \mathbb{R}$, all the three sums are finite.

The wavelet decompositions (6.8) and (6.9) can be used to build linear filters. Hence, it is very desirable to have the property of linear phase, or at least generalized linear phase, for the dual bases $\tilde{\varphi}$ and $\tilde{\eta}$. We have the following result.

THEOREM 6.2. *Let $\varphi \in \Phi$ and $\psi \in \Psi_p$ be both minimally supported such that the defining two-scale sequence $\{p_n^\varphi\}$ of φ is real-valued. Then*

(i) *if $\{p_n^\varphi\}$ has generalized linear phase, both the duals $\tilde{\varphi}$ and $\tilde{\psi}$ also have generalized linear phases; and*

(ii) *if $\{p_n^\varphi\}$ has linear phase, both the duals $\tilde{\varphi}$ and $\tilde{\psi}$ have linear phases.*

Proof. We first prove this theorem for $\tilde{\varphi}$. By Lemmas 5.2–5.4, since $z^{-k_\varphi}\Pi_\varphi(z) > 0$ on $|z| = 1$, the sequence $\{2\bar{a}_n\}$ with symbol $\overline{G_\varphi(z)}$ defined in (4.11), which is the two-scale sequence of $\tilde{\varphi}$, has generalized linear phase (or linear phase), provided that $\{p_n^\varphi\}$ has this property. Hence, it follows from Remark 5.6 and Theorems 5.1 and 5.2, that if $\{p_n^\varphi\}$ is of generalized linear phase, so is $\tilde{\varphi}$; and if $\{p_n^\varphi\}$ is of linear phase, so is $\tilde{\varphi}$.

To investigate the phase property of $\tilde{\psi}$, we note from Theorem 6.1 that its two-scale sequence is given by $\{2\bar{b}_n\}$ whose symbol is $\overline{H_\psi(z)}$, where

$$H_\psi(z) = -z^{-2N_\varphi + 2k_\varphi + 1} \frac{P_\varphi(-z)\,\lambda_\varphi(z^2)}{\Pi_\varphi(z^2)} \tag{6.10}$$

(cf. (4.12) with $w_\psi = 1$). Assume that $\{p_n^\varphi\}$ has linear phase. Then since the polynomial λ_φ is zero-free on $|z| = 1$ (cf. Remark 4.3), following the proof of Theorem 5.3, we see that

$$z^{-\deg \lambda_\varphi} \lambda_\varphi(z^2) > 0, \qquad |z| = 1.$$

Next, recall that $N_\varphi =: 2N'_\varphi$ is even (cf. Lemma 5.4) and $P_\varphi(e^{i\omega}) = \pm |P_\varphi(e^{i\omega})|\, e^{iN'_\varphi\omega}$ (cf. Remark 5.5). Hence, $P_\varphi(-e^{i\omega}) = \pm |P_\varphi(-e^{i\omega})| \times e^{iN'_\varphi\omega} e^{iN'_\varphi\pi}$ where $e^{iN'_\varphi\pi} = \pm 1$, so that $\{(-1)^n p_n^\varphi\}$ also has linear phase. Combining these two observations and the fact that $z^{-2k_\varphi}\Pi_\varphi(z^2) > 0$ for $|z| = 1$, we may conclude from (6.10) that $\{2b_n\}$, and hence $\{2\tilde{b}_n\}$, also has linear phases. So, by Theorem 5.3, $\tilde{\psi}$ has linear phase. The proof for the generalized linear phase of $\tilde{\psi}$ is similar. ∎

Remark 6.5. In order to achieve the generalized linear phase property while maintaining the compact supports, Cohen [8] (cf. also, [11]) introduced the notion of biorthogonal bases. While there seems to be certain similarities between the dual bases in this paper and the biorthogonal bases in [8], they are really quite different. Let us briefly discuss what biorthogonal bases are meant to be. Let $\phi \in \Phi$, set

$$h_n = \frac{1}{\sqrt{2}}\, p_n^\varphi, \tag{6.11}$$

and choose a finite sequence $\{\tilde{h}_n\}$ that satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \sum_{n \in \mathbb{Z}} h_n \tilde{h}_{n+2l} = \delta_{l,0},\ l \in \mathbb{Z}; \text{ and} \\ &\text{(ii)} \quad \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\omega) \end{aligned} \tag{6.12}$$

converges uniformly on all compact subsets of $\mathbb{R}$, where

$$\tilde{m}_0(\omega) := \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{h}_n e^{-in\omega}. \tag{6.13}$$

Let $\tilde{\phi}_c \in L^2$ whose Fourier transform $\hat{\tilde{\phi}}_c(\omega)$ is defined by the infinite product (6.12). Then it was proved in [8] that $\tilde{\phi}_c$ has compact support and is a "dual" of ϕ in the sense

$$\langle \phi(\cdot - m), \tilde{\phi}_c(\cdot - n) \rangle = \delta_{m,n}, \qquad m, n \in \mathbb{Z}. \tag{6.14}$$

We note, however, that although the "duality" condition (6.14) is the same as ours, this "dual" $\tilde{\phi}_c$ is *not* in V_0, in general. With both sequences $\{h_n\}$ and $\{\tilde{h}_n\}$ already determined, the "wavelet" ψ_c and its "dual" $\tilde{\psi}_c$ are now defined by

$$\psi_c(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} (-1)^n \tilde{h}_{-n+1}\, \phi(2x - n) \tag{6.15}$$

and

$$\tilde{\psi}_c(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} (-1)^n h_{-n+1} \tilde{\phi}(2x - n). \tag{6.16}$$

Let us pause for a moment to remark that this approach is also different from ours in that we determine the duals $\tilde{\varphi}$ and $\tilde{\psi}$ from φ and ψ, while Cohen determines ψ_c and $\tilde{\psi}_c$ from ϕ and $\tilde{\phi}_c$. The "wavelet" ψ_c and its dual $\tilde{\psi}_c$ of Cohen's also satisfy the same "duality" properties as ours, namely

$$\begin{aligned} \langle \psi_{c;k,j}, \tilde{\psi}_{c;l,m} \rangle &= 2^{-k} \delta_{k,l} \delta_{j,m} \\ \langle \psi_{k,j}, \tilde{\psi}_{l,m} \rangle &= 2^{-k} \delta_{k,l} \delta_{j,m}, \end{aligned} \tag{6.17}$$

where we have used the same dilation and translation definition as (1.13) for both ψ_c and $\tilde{\psi}_c$. However, again the similarity ends here. Indeed, if W_k^c and $\tilde{W}_k^c$ denote, as usual, the closures of the linear spans of $\{\psi_{c;k,j}: j \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{c;k,j}: j \in \mathbb{Z}\}$, respectively, then not only are W_k^c, $\tilde{W}_k^c$ different, they are also different from our orthogonal wavelet spaces W_k. It is interesting to point out, however, that each of $\{W_k^c\}$ and $\{\tilde{W}_k^c\}$, $k \in \mathbb{Z}$, provides a *direct-sum* decomposition of L^2, which is non-orthogonal, in general. For more details, see [8, 11].

7. Final Remarks

In this paper, the multiresolution analysis generator $\phi \in \Phi$ is defined by its two-scale formula (1.4), or equivalently, its Fourier transform is defined by

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} P_\phi(e^{-i(\omega/2^j)}). \tag{7.1}$$

Hence, the existence of a nontrivial solution of (1.4) is equivalent to the convergence of the infinite product in (7.1). This problem has been thoroughly investigated in [9, 12, 16, 2]. In particular, Daubechies [9] proved that if

$$P_\phi(z) = \left(\frac{1+z}{2}\right)^m M(z), \qquad m \geqslant 1, \tag{7.2}$$

where $M \in \pi$ satisfies $M(1) = 1$ and

$$\sup_{\omega \in \mathbb{R}} |M(e^{-i(\omega/2)}) \cdots M(e^{-i(\omega/2^l)})| < 3^{l(m-1)} \tag{7.3}$$

for some positive integer l, then the infinite product (7.1) converges.

Another useful condition for the existence of a nontrivial solution to (1.4) is

$$\inf_{|\omega| < \omega/2} |P_\phi(e^{i\omega})| > 0; \qquad \text{and}$$
$$\sup_{n \geq 0} \int_{|\omega| < 2^n\pi} \left| \prod_{j=1}^{s} P_\phi(e^{-i(\omega \cdot 2^j)}) \right| d\omega < \infty \tag{7.4}$$

given by Meyer [16].

The importance of the (generalized) B-spline $\varphi \in \Phi$ is that it provides the information on the order of approximation from the multiresolution analysis spaces $\{V_k\}$ and allows us to construct approximation and interpolation formulas that guarantee this order of approximation (cf. [19, 21, 3]). On the other hand, the importance of the B-wavelet is its orthogonal property which enables us to analyse the best approximants. In the following, we give a summary of the equivalence of some of these properties.

THEOREM 7.1. *Let $\varphi \in \Phi$ be the (generalized) B-spline that generates a given multiresolution analysis, and let $\eta \in \Psi_p$ be arbitrarily chosen. Then the following statements are equivalent:*

(i) *The order of approximation of φ is m, in the sense that*

$$\inf_{g \in V_n} \|f - g\| = O\left(\left(\frac{1}{2^n}\right)^m\right)$$

for all $f \in C^m \cap L^2$.

(ii) $D^j\hat{\phi}(2\pi l) = 0$, $l \in \mathbb{Z} \setminus \{0\}$, $j = 0, ..., m-1$.

(iii) *The commutator order of φ is m in the sense that*

$$[g \mid \varphi] := \sum_{j \in \mathbb{Z}} g(j)\, \varphi(\cdot - j) - \sum_{j \in \mathbb{Z}} \varphi(j)\, g(\cdot - j)$$

is identically zero for all polynomials g of degree $\leqslant m-1$.

(iv) $P_\varphi(z) = ((1+z)/2)^m M(z)$ *for some* $M \in \pi$ *with* $M(1) = 1$.

(v) $\sum_{n=0}^{N_\varphi} (-1)^n n^j p_n^\varphi = 0$, $j = 0, ..., m-1$.

(vi) $\int_{-\infty}^{\infty} x^j \eta(x)\, dx = 0$, $j = 0, ..., m-1$.

For a proof of this result, see a combination of [19, 21, 12, 3, 20].

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